

### Practice Problems: Final Exam, Set 1

We have chosen these problems because we think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the exam will resemble these problems in any way whatsoever. Remember that on exams you will have to supply evidence for your conclusions and explain why your answers are appropriate.

1. In this question, you will set up equations for two periodic functions. Some of the data that you will need to do this is given in Table 1 (below)<sup>1</sup>. In Table 30, January 1 is “Day 0” of the year.

Day of the Year	Hours of Daylight	Day of the Year	Hours of Daylight
0	9.1	210	14.6
30	9.9	240	13.3
60	11.2	270	11.9
90	12.7	300	10.6
120	14	330	9.5
150	15	360	9.1
180	15.3		

Table 1: Daylight hours during 1998 for Cambridge, MA.

- (a) Use the data in Table 1 to plot a graph showing Hours of Daylight versus Day for Cambridge, MA. Calculate the period, midline and amplitude, and show how these quantities appear on your graph. Find an equation for the number of hours of daylight as a function of the day for Cambridge, MA.
- (b) Figure 1<sup>2</sup> shows the location of the city of Cambridge, New Zealand. Use the data given in Table 1 to sketch a plausible graph showing Hours of Daylight versus Day for Cambridge, New Zealand. Find an equation for the number of hours of daylight as a function of the day for Cambridge, New Zealand.

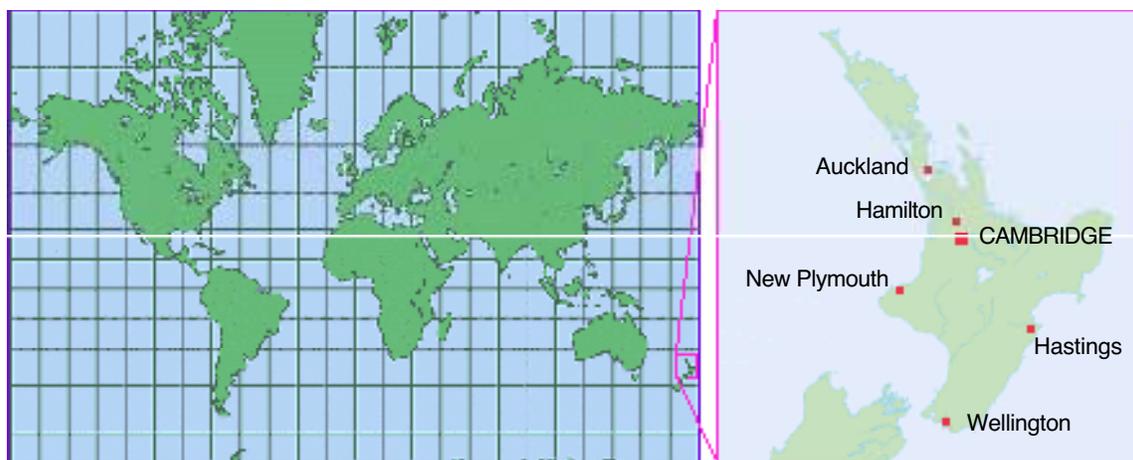


Figure 1: Location of Cambridge, New Zealand. Note that the country of New Zealand is located in the Southern Hemisphere.

<sup>1</sup> Source of data: The Texas Education Network, Texas Essential Knowledge and Skills, Mathematics Module: Oscillations.

<sup>2</sup> Image sources: <http://www.kart-a-bag.com> and <http://www.maritimeheritage.org/>

2. Agricultural production in the United States can be reasonably well approximated by the sum of a linear function and a periodic function (see Figure 2). A formula for this function,  $A(T)$  is given below, in which  $T$  represents the number of years since 1989, and  $A(T)$  represents the rate of US agricultural production in units of millions of metric tons per year.

$$A(T) = 173.5245 + 2.592 \cdot T + 15.2895 \cdot \cos\left(\frac{2\pi}{8} \cdot T\right).$$

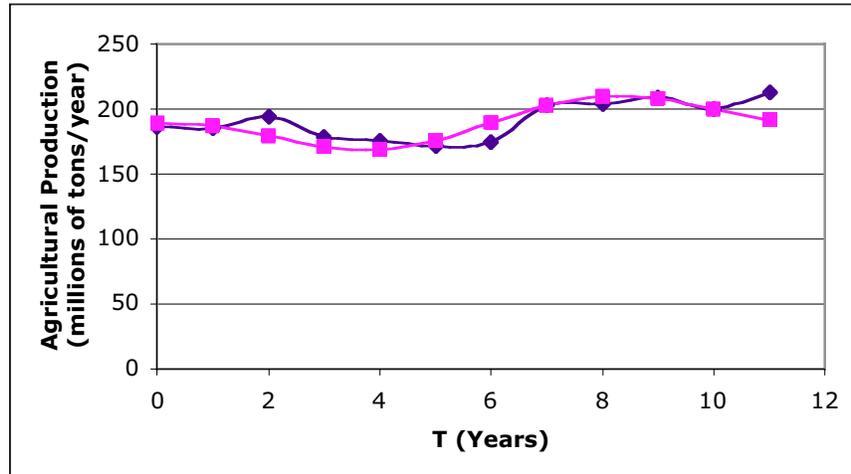
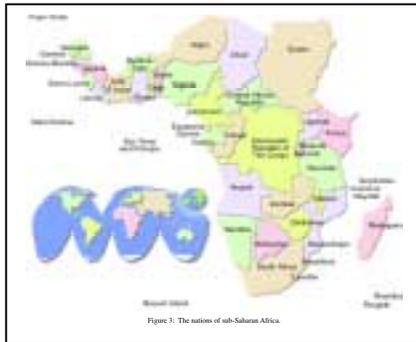


Figure 2: Plot of U.S. Agricultural Production (blue curve) and function  $A(T)$  (purple curve), 1989-2000.



Not every country in the world is as fortunate as the United States and many do not enjoy the abundance of food that we enjoy here. (Remember the homework assignment on SlimFast<sup>®</sup> milkshakes? We have so much food that Americans spend \$107 million *every day* on weight-loss products and diets.) For example, many African are currently unable to produce enough food to meet even the base survival needs of their population, or will soon be unable to meet the survival needs of their population. The U.S. Department of Agriculture estimates that by 2010 (eight years away) more than 60% of the 37 countries that make up sub-Saharan Africa (see Figure 3<sup>3</sup>) will be unable to meet the basic survival needs of their population<sup>4</sup>.

In the Biblical story of Genesis 41:1-37 (you can download this separately and read it if you wish to), the Pharaoh of Egypt has a prophetic dream in which he sees seven well-fed cattle, followed by seven “lean-fleshed and ill-favored” cattle which consumed the well-fed cattle. The Pharaoh’s butler has a Hebrew servant named Joseph who interprets the dream to mean that Egypt will enjoy seven years of abundant harvests followed by seven years of drought. Joseph advises the Pharaoh to appoint overseers to collect 20% of the grain harvested in the years of plenty, and to store this surplus to prevent a famine in the seven years of drought.

<sup>3</sup> Image source: <http://www.nbij.gov/>

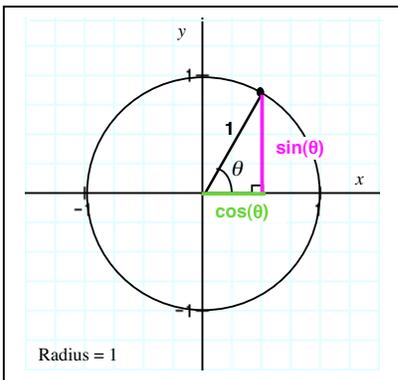
<sup>4</sup> Source: U.S. Department of Agriculture, Economic Research Service. “Global Food Security Overview.” Food Security Assessment GFA-12, December 2000. You can also look up a calculation that you did for the sub-Saharan nation of Kenya in Math Xa Homework Assignment #20, (see [http://www.courses.fas.harvard.edu/~mathxa/homework/assignments/xa\\_hw20.pdf](http://www.courses.fas.harvard.edu/~mathxa/homework/assignments/xa_hw20.pdf) and [http://www.courses.fas.harvard.edu/~mathxa/homework/assignments/xa\\_hw20\\_sol.pdf](http://www.courses.fas.harvard.edu/~mathxa/homework/assignments/xa_hw20_sol.pdf)) in which you calculated that by 2007-2008 (five or six years away) Kenya will no longer be able to produce enough maize (by far the most important food crop) for its entire population.

The plan that Joseph suggested to Pharaoh is a simple idea – an idea practiced by such animals as the Black-capped chickadee (*Poecile atricapillus*) – that excess food should be stored in times of plenty in a food cache and retrieved in times of need. The difficult part – as governments and humanitarian relief organizations generally do not have prophetic dreams on which to base their long-term plans – is predicting when the times of plenty and the times of need are likely to occur. In this question, you will use calculus and the function  $A(T)$  to do this.

- (a) Find equations for the first and second derivatives of  $A(T)$ . Use the equation for the first derivative to locate *all* critical points (there are infinitely many of them) of the function  $A(T)$ .
- (b) Use the equations that you found in Part (a) to determine when the agricultural production of the United States will reach:
- A local maximum, or,
  - A local minimum,

and indicate which years give a local maximum, and which years give a local minimum of agricultural production.

**NOTE:** You need to find and describe *all* years when the agricultural production of the United States has a local maximum and *all* years when the agricultural production of the United States has a local minimum. There are infinitely many such points, so don't try writing them all out one-by-one.



3. In this question, all that you have to do is to find equations for anti-derivatives. Don't forget to add the "+C" to the end!

- (a) Find an equation for the anti-derivative:

$$\int \frac{\sin(\theta)}{\sqrt{1 - \sin^2(\theta)}} \cdot d\theta.$$

**NOTE:** One consequence of combining the Pythagorean Theorem with the unit circle definition of the sine and cosine is the useful relationship:

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

- (b) Find an equation for the anti-derivative:  $\int \tan(\theta) \cdot \cos(\theta) \cdot d\theta$

**NOTE:** The second anti-derivative has a definite "trick" to it. One calculus fact that might help you is:  $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ . Using this expression, the anti-derivative of  $\tan(\theta)$  is given by a  $u$ -substitution with  $u = \cos(\theta)$ .

$$\int \tan(\theta) \cdot d\theta = \int \frac{\sin(\theta)}{\cos(\theta)} \cdot d\theta = \int \frac{-1}{u} \cdot du = -\ln(u) + C = -\ln(\cos(\theta)) + C.$$

4. In this problem, the idea is to find symbolic solutions for differential equations using the technique of *Separation of Variables*. These problems are intended to help you to get used to performing the technique of *Separation of Variables*. There are no trick questions – all that you have to do in each problem is to find a formula for the function  $y(t)$ , given a differential equation and an initial condition.

- (a) **Differential Equation:**  $y'(t) = \frac{1}{2} \cdot y(t)$ .  
**Initial value:**  $y(0) = 5$ .
- (b) **Differential Equation:**  $y'(t) = 4 \cdot y(t)$ .  
**Initial value:**  $y(0) = 64$ .
- (c) **Differential Equation:**  $y'(t) = \frac{1}{2} \cdot [y(t) - 1]$ .  
**Initial value:**  $y(0) = 9$ .
- (d) **Differential Equation:**  $y'(t) = 3 \cdot [1 - y(t)]$ .  
**Initial value:**  $y(0) = 10$ .
- (e) **Differential Equation:**  $y'(t) = 2 \cdot y(t) - 8$ .  
**Initial value:**  $y(0) = 7$ .

**REMEMBER:** If you are not sure about your answer, you can always check it by seeing if the formula that you have derived satisfies both the differential equation and initial condition.

5. In this question, you will use the concept of average value to calculate the average depth of the Pacific Ocean. Figure 4<sup>5</sup> shows a cross-section of the Pacific Ocean stretching from Japan on the left to California on the right. (The region of tall, narrow spikes near the center of Figure 4 is Hawaii.)

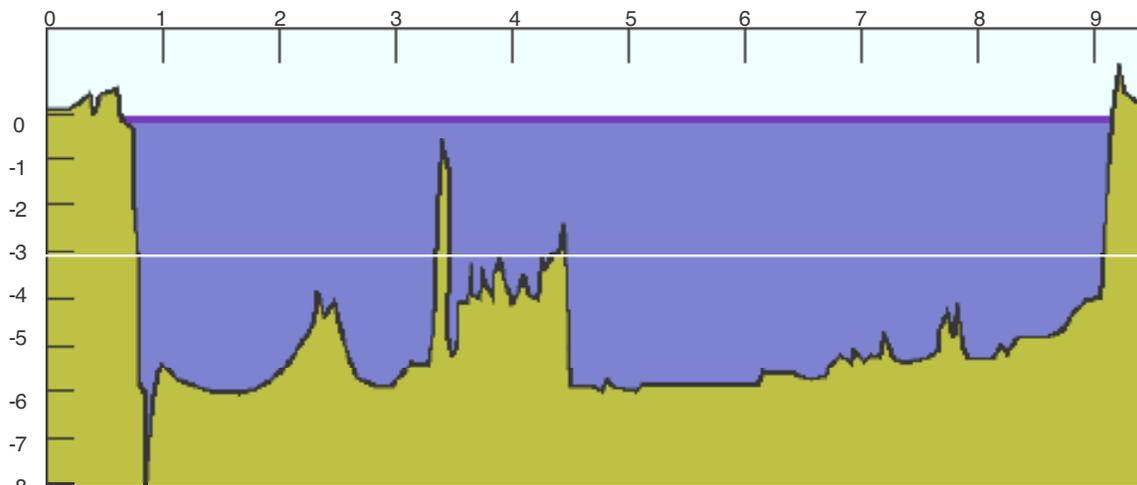


Figure 4: Cross-section of the Pacific Ocean. The numbers on the vertical axis are the depth in units of kilometers (km) and the numbers on the horizontal axis are the distance from Japan in units of thousands of kilometers.

<sup>5</sup> Image source: <http://geosun1.sjsu.edu/>

(a) Using the variables:

$x$  = Distance from Japan (in units of thousands of kilometers)

$y$  = Depth of Pacific Ocean (in units of kilometers)

the shape of the ocean floor shown in Figure 4 can be approximated by the equation:

$$y = p(x) = 0.048 \cdot x^4 - 0.9 \cdot x^3 + 5.79 \cdot x^2 - 14.89 \cdot x + 7.69.$$

Find an equation for an anti-derivative,  $P(x)$ , of  $p(x)$ . Your equation for  $P(x)$  should include one unspecified constant (i.e. remember the “+ $C$ ”!). If you were to evaluate the quantity

$$P(9) - P(0.7),$$

which of the following would you have calculated:

- The light blue area from Figure 4?
- The dark blue area from Figure 4?
- The tan area from Figure 4?

Indicate which of these areas is represented by the quantity  $P(9) - P(0.7)$  as part of your answer to this problem.

(b) What is the average depth of the Pacific Ocean (in units of kilometers)? Show any relevant work that you used to determine your answer.