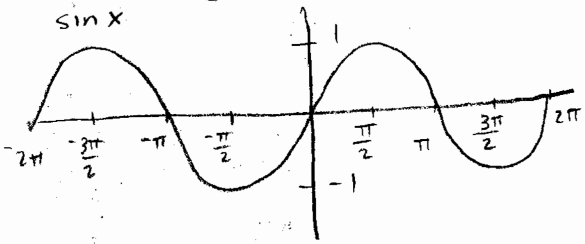
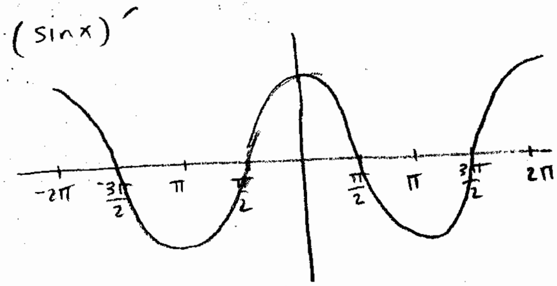
Ch. 21

Ch. 21.1 Derivative of sin x

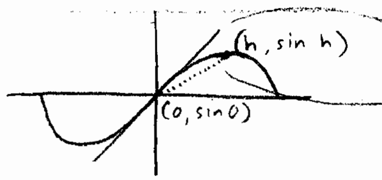
From the graph of sin x we can guess the shape of the graph of $(\sin x)'$





- where sin x is inc, $(\sin x)'$ is pos
- where sin x is dec, $(\sin x)'$ is neg
- where sin x has a local max/min $(\sin x)'$ is 0.
- this looks like cos x, but we're not sure so...

if we want to estimate the slope at $x=0$, we can use a secant line from $x=0$ to a nearby point $(h, \sin h)$



→ tangent line to sin x at $x=0$

→ secant line between (0,0) and (h, sin h)

the slope of the secant is $\frac{\sin(h) - 0}{h - 0} = \frac{\sin h}{h}$

in Ch 21.2 we learn that $\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$

meaning that the slope of the secant goes to 1 as it approaches the tange
meaning! that the deriv. of sin x at $x=0$ is 1. (which is good b/c $\cos 0 = 1$)

Ch 21.2 Differentiating sin x + cos x

All of this becomes important when trying to find the
derivs of sin x and cos x.

- From the proofs in Ch 21.2, we learn that : $\boxed{\frac{d}{dx} \sin x = \cos x}$
  ... which is the conclusion 21.1 was leading us to.
- using the fact that $\cos x = \sin\left(\frac{\pi}{2} + x\right)$ we get $\boxed{\frac{d}{dx} \cos x = -\sin x}$
- using these two deriv.s, we can find the derivs of other trig functions. $\boxed{\frac{d}{dx} \tan x = \sec^2 x}$

$\boxed{\frac{d}{dx} \sec x = \sec x \tan x}$

Ch 21.3 optimization using Trig

General strategy for solving optimization problems:

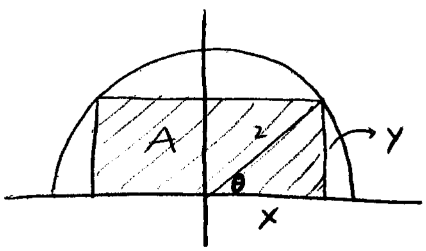
① **Draw a picture** - label all parts that are important and any known + unknown info.

② **Write an equation** - for the quantity whose max or min you want
- try to get the equation in terms of one variable for ex., $y = f(x)$
- this may require some algebra or substitution
- note the domain of y and x (e.g. can't have negative distances)

③ **Test all critical pts and all end pts.** - list values of $f(x)$ at both crit pts (where $f'(x) = 0$) and endpts to find max. or min.

Ex: Imagine you are trying to maximize a rectangle inscribed in a semicircle of radius 2. What angle θ gives the largest area A?

① Draw a picture



$\frac{x}{2} = \cos\theta \qquad \frac{y}{2} = \sin\theta$

$x = 2\cos\theta \qquad y = 2\sin\theta$

② Write an eq for Area

$A = (2x)y$

- try to get in terms of 1 var. using subst.

$A = 2(2\cos\theta)(2\sin\theta)$

$A = 8\sin\theta\cos\theta$

- note domain + range

$0 \le \theta \le \frac{\pi}{2} \qquad A \ge 0$

③ Test crit. pts + end pts

- crit. pts where $\frac{dA}{d\theta} = 0$

$\frac{dA}{d\theta} = 8(\sin\theta \cdot -\sin\theta + \cos\theta \cdot \cos\theta)$

$\frac{dA}{d\theta} = 8(\cos^2\theta - \sin^2\theta)$

$0 = 8(\cos^2\theta - \sin^2\theta)$

$\cos^2\theta = \sin^2\theta$

$\cos\theta = \pm\sin\theta$

- now use our knowledge of trig values to say $\cos\theta = \pm\sin\theta$ where $\theta = \frac{\pi}{4}$ (since $0 \le \theta \le \frac{\pi}{2}$)

$\theta = \frac{\pi}{4}$

- crit pt

$A\left(\frac{\pi}{4}\right) = 8\left(\sin\frac{\pi}{4}\cos\frac{\pi}{4}\right) = 8\left(\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}\right) = \boxed{4}$ ✓

4 is the max area

- end pts

$A(0) = 8\sin 0 \cos 0 = 0$

$A\left(\frac{\pi}{2}\right) = 8\sin\frac{\pi}{2}\cos\frac{\pi}{2} = 0$

$\theta = \frac{\pi}{4}$

Ch 21.3 Related Rates using trig

- General strategy very similar to optimization problems
- instead of asking for a max or min value but a value or rate at a specific pt (usually)

① Draw a picture, name variables and constants

② Write down numerical information

③ Write down what you're asked to find (usually a rate like $\frac{dx}{dt}$)

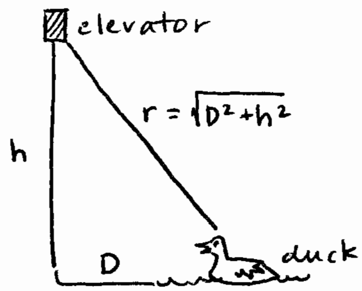
④ Write an equation that relates the variables

⑤ Differentiate · solve for the rate you want in terms of known values

⑥ Evaluate · to find the rate you want

Ex: Ch 21.3 # 23

① Draw picture



② Write down numerical info

$$\frac{dh}{dt} = 10$$

$$\frac{dD}{dt} = -5 \quad \text{(neg since D is dec.)}$$

③ Asked to find $\boxed{\frac{dr}{dt}} = ?$

when $h = 100$ $D = 200$ $r = \sqrt{200^2+100^2} = \sqrt{50000}$

~~[illegible]~~:

Geom: ④ Write an eq.

$$r^2 = h^2 + D^2$$

⑤ Differentiate

$$2r \cdot \frac{dr}{dt} = 2h\frac{dh}{dt} + 2D\frac{dD}{dt}$$

⑥ Evaluate

$$2(\sqrt{50{,}000})\frac{dr}{dt} = 2 \cdot 100 \cdot 10 - 2 \cdot 200(-5)$$

$$\boxed{\frac{dr}{dt} = 0}$$

Trig ③ Part b asks to find $\boxed{\frac{d\theta}{dt}} = ?$

④ Write an eq.

$$\tan\theta = \frac{h}{D}$$

⑤ Differentiate

$$\sec^2\theta \frac{d\theta}{dt} = \frac{D\left(\frac{dh}{dt}\right) - h\left(\frac{dD}{dt}\right)}{D^2}$$

⑥ Evaluate · 1st we find $\sec\theta$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{D} = \frac{\sqrt{50000}}{200}$$

$$\left(\frac{\sqrt{50{,}000}}{200}\right)^2 \frac{d\theta}{dt} = \frac{200(10) - 100(-5)}{200^2}$$

$$\boxed{\frac{d\theta}{dt} = .05 \frac{\text{rad}}{\text{sec}}}$$

- answer is in radians

ch 21.4 Deriv. of Inverse Trig Functions

- Remember that $y = \sin^{-1} x$ is equivalent to $x = \sin y$
- but remember also that inverse trig. functions have restricted domains and ranges.

$f(x) = \sin^{-1} x \qquad \frac{-\pi}{2} \le x \le \frac{\pi}{2}$

$f(x) = \cos^{-1} x \qquad 0 \le x \le \pi$

$f(x) = \tan^{-1} x \qquad \frac{-\pi}{2} \le \tan^{-1} x \le \frac{\pi}{2}$

$$\boxed{\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}}$$

Ch. 21 Common Mistakes

Math Xb
3/24/04

21.1 #3

They want you to estimate $f'(2)$ and $f''(2)$ using the def. of the derivative.
So, for ex:

$$f'(2) \cong \frac{f(1.999) - f(2)}{1.999 - 2} = -0.416 \qquad f''(2) \cong \frac{f'(1.999) - f'(2)}{1.999 - 2} = -0.5$$

21.2 #10

$$\frac{d}{dx}\left[\frac{1}{\sin^3(\cos(2x))}\right] = \frac{(\sin^3(\cos 2x) \cdot 0 - 1 \cdot \frac{d}{dx}\left[\sin^3(\cos 2x)\right]}{[\sin^3(\cos 2x)]^2}$$

→ Quotient Rule
→ Notice that you take the $\sin^3$ of $\cos 2x$, it's not $\sin^3 x \cdot \cos 2x$

$$= \frac{-\left[3\sin^2(\cos 2x) \cdot \frac{d}{dx}\left[\sin(\cos 2x)\right]\right]}{[\sin^3(\cos 2x)]^2}$$

→ Power Rule, then Chain Rule
→ take derivative of $\sin(\cos 2x)$ b/c that is the expression being cubed.

$$= \frac{-3\sin^2(\cancel{\cos 2x}) \cdot \left[\cos(\cos 2x) \cdot (-\sin 2x) \cdot (2)\right]}{[\sin^6(\cos 2x)}$$

→ this is chain rule applied 3 times
→ denominator becomes $\sin^6$ (exp rules)

$$= \frac{6\cos(\cos 2x) \cdot (\sin 2x)}{\sin^4(\cos 2x)}$$

21.3 #1

Note that the domain is not restricted, so you have to consider all max + min, and all intervals for $x \in (-\infty, \infty)$

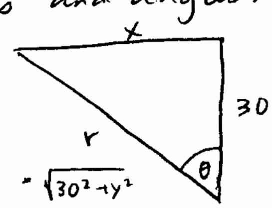
21.3 #5 x is restricted here: $x \in [0, 2\pi]$

$f'(x) = -2\sin 2x + 2\sin x = -2(2\sin x \cos x) + 2\sin x = 2\sin x(-2\cos x + 1)$

↳ $\sin x = 0$ or $\cos x = \frac{1}{2}$ ⇒ $x = 0, \pi, 2\pi, \frac{\pi}{3}, \frac{5\pi}{3}$

21.3 #8 This is a related rates problem using trigonometry

Because we're trying to find $\frac{d\theta}{dt}$, we first must find an expression relating distances and angles. Let's try $\tan\theta$ (we can also use $\cos\theta$ or $\sin\theta$)



do you see why its easier to use $\tan\theta$?
we know 30 stays constant and x = the opposite side.
whereas if we chose $\sin\theta$, we'd have $\sin\theta = \frac{x}{r}$ → 2 unknown

① So 1st we write our equation, only plugging in specific numbers if they are constant
$\tan\theta = \frac{x}{30}$ ② next we differentiate: $\sec^2\theta \frac{d\theta}{dt} = \frac{1}{30} \cdot \frac{dx}{dt}$

also $\frac{d\theta}{dt}$ should be in radians

③ now we plug in the values at the specific point we're interested in:
[illegible] 30 $r = \sqrt{3400}$ so $\cos\theta = \frac{50}{\sqrt{3400}}$ ⇒ $\left(\frac{\sqrt{3406}}{50}\right)^2 \frac{d\theta}{dt} = \frac{1}{30}(-46)$ ⇒ $\frac{d\theta}{dt} = \boxed{-.405 \frac{rad}{s}}$

21.3 #12

a) fast and easy way to do the problem:

$x = \left[\frac{v_0^2}{g} \cdot \sin 2\theta\right]$ will have a max when $(\sin 2\theta)$ is greatest (since v_0 and g are just unchanging constants. So what's the max of $(\sin 2\theta)$?

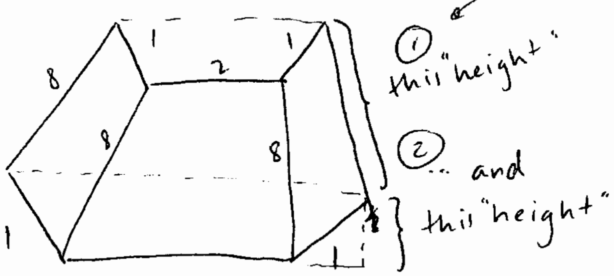
The greatest $\sin 2\theta$ can be is 1. So we can say

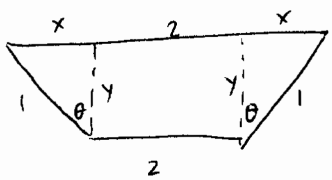
$$x_{max} = \frac{v_0^2}{g}(1) = \boxed{\frac{v_0^2}{g}}$$

b) be careful of unit conversions!

21.3 #14 - optimization using Trig

Most people got tripped up because there are two heights in the problem:





so $V = (\text{Area of Base})(\text{Height})$

if we look at it like a cylinder w/ a trapezoid base:  Height

but the Height of the cylinder is constant = 8

so $V = (\text{Area of Trap})(8)$

→ The Area of my Trap $= \frac{1}{2}\left[2 + (2+2x)\right][y]$

→ the important thing to notice is that:

$\frac{x}{1} = \sin\theta \qquad \frac{y}{1} = \cos\theta$

→ so plugging back: $V = \left[\frac{1}{2}(4 + 2\sin\theta)\cdot\cos\theta\right]\cdot 8$

$V = 16\cos\theta + [8\sin\theta\cos\theta]$

we want to find the angle θ that gives us the greatest V.

so we find $\frac{dV}{d\theta} = 0$ to find maxima of V.

→ and differentiating: $\frac{dV}{d\theta} = -16\sin\theta + [-8\sin^2\theta + 8\cos^2\theta]$

→ subbing trig IDs to solve ($\cos^2\theta = 1 - \sin^2\theta$)

$= -16\sin\theta + [-8\sin^2\theta + 8 - 8\sin^2\theta]$

$= -16\sin^2\theta - 16\sin\theta + 8$

→ set $\frac{dV}{d\theta} = 0$ $\qquad 0 = -2\sin^2\theta - 2\sin\theta + 1$

→ solve for θ $\qquad \sin\theta = \frac{2 \pm \sqrt{12}}{-4}$

→ reject neg $\sin\theta$ $\qquad \sin\theta = \frac{2 - \sqrt{12}}{-4} \Rightarrow \boxed{\theta = 0.375\text{ rad}}$

→ we should also check boundary points and make sure our θ is actually a max

$V(0) = 1 \cdot 2 \cdot 8 = 16 \quad V\left(\frac{\pi}{2}\right) = 0 \quad V(.375) = 16\cos(.375) + 8\cos(.375)\sin(.375)$