Chapter 22

The Definite Integral:

a. how an "amount" function is recovered from a "rate" function. ex. finding distance if the speed function is known (helps us find net change in amount)

b. how to calculate the area between the graph of a function & the axis.

- The relationship between a & b is easy to see when the rate function is constant.

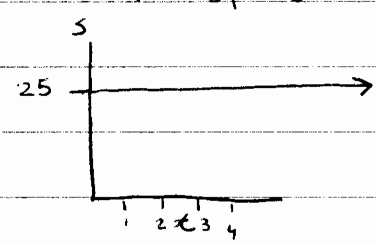
(ex) I drive at 25 miles an hour for 4 hours, how far have I gone?

a. we know that rate $= \frac{\text{distance}}{\text{time}}$, so distance = rate · time

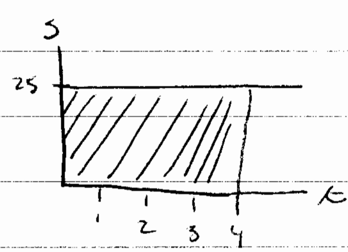
distance $= (25 \frac{\text{miles}}{\text{hour}})(4 \text{hours}) = 100$

b. or we could think graphically & say that $s(t)$ = speed at time t so $s(t) = 25$

so



Now, to find the distance traveled in 4 hours we find the area under $s(t)$ for 4 hours



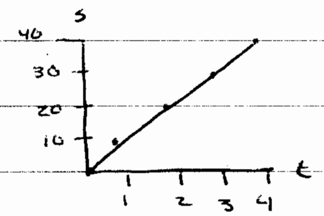
$= 4 \times 25 =$ (length × width) $= 100$

So we get the same answer!

- The relationship still Applies when the rate of change is Not constant

ex) This time I drive with speed, $s(t) = 10t$ ~~for~~ from $t=0$ to $t=4$.

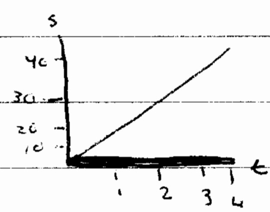
Graphically,



From the given equation, $s(t)=10t$, we know that I traveled at between 0 mph, $s(0)=0$, & 40mph $s(4)=40$

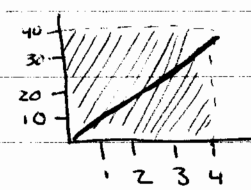
So, at the least I've traveled $0 \times 4 = 0$ miles

& at the most I've traveled $40 \times 4 = 160$ miles



= 0 area

or



$40 \times 4 = 160$ area

to get a better estimate we can measure speed at smaller intervals, say every hour

so,

t	$s(t)$
0	0
1	10
2	20
3	30

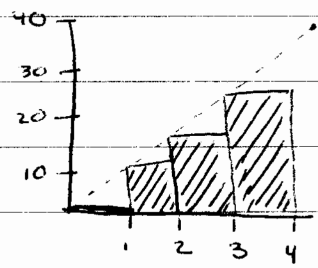
In the first hour I travel at a minimum speed of 0mph, In the 2nd hour my minimum speed is 10mph, The minimum speed for the 3rd & 4th hour are 20mph & 3mph

So the estimate of minimum distance becomes

$$d = (0\text{mph})(1\text{h}) + (10\text{mph})(1\text{h}) + (20\text{mph})(1\text{h}) + (30\text{mph})(1\text{h})$$

$$\boxed{d = 60\text{m}}$$

or

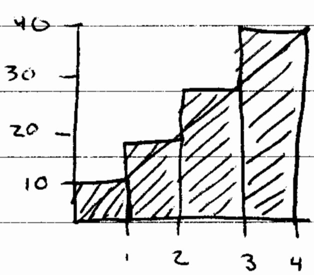


which is a better estimate of the area under the graph & of the distance traveled.

In this example we have found L_4 of $S(t) = 10t$.
That is a left-hand sum with four divisions.
Left hand sums take the height of the interval's left end-point.

Q. For practice find R_4 of $S(t) = 10t$

A.



$(10mph)(1h) + (20mph)(1h) + 30(mph)(1m) + (40mph)(1h)$
$= 100m$

The right hand sum uses the same intervals, but takes the height of the right end point

Two things to notice:

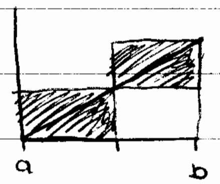
1. each term is a product of a width & a height. The width is the segment width & the height is the functional value at one of the end points
2. All of the terms in L_4 & R_4 are the same except for the first term of L_4 & the last of R_4

For $F(x)$: on $[a, b]$

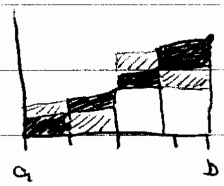
$\therefore |R_n - L_n| = [\text{width}(F(b)) - \text{width}(F(a))]$ width $= \frac{b-a}{n}$

so $|R_n - L_n| = \frac{b-a}{n}(F(b) - F(a))$ Notice, as N increases $|R_n - L_n|$ decreases

This is easy to see on a graph because, as the difference between R_n & L_n decrease we are getting closer to the actual area.



Here $|R_2 - L_2|$ has been shaded.



Here $|R_4 - L_4|$ has been shaded darkly & the additional area for $|R_2 - L_2|$ has been shaded lightly.

* $\lim_{n\to\infty} |R_n - L_n| = 0$ $\therefore$ $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n =$ area under curve
- by squeeze theorem.

* The Definite Integral of $f(x)$ from a to b is written $\int_a^b f(x)\,dx$ & means the signed area under the graph of $f(x)$ & the net change in amount of $f(x)$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$$

$f(x)$ = the integrand

a & b = limits of integration.

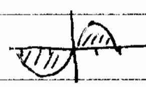
Signed Areas:

- areas above the axis are positive (when moving from left to right)
- areas below the axis are negative (when moving from left to right)
- The opposite is true when moving right to left

Q. Determine the sign of the following integrals:

a) $\int_{-\pi}^{\pi} \sin(x)\,dx$ c) $\int_{1}^{0} x^2\,dx$

b) $\int_{-\pi/2}^{\pi/2} \cos(x)\,dx$ d) $\int_{0}^{1} x^2\,dx$

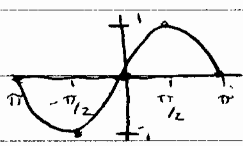
A. a)  $= 0$ no sign c)  ⊖ moving Right to left

b)  $=$ ⊕ D)  ⊕

Q. Place the following in Ascending order (some may be equal)

A) $\int_{-\pi}^{\pi} \sin(x)\,dx$ C) $\int_{-\pi}^{0} \sin(x)\,dx$ E) $\int_{0}^{-\pi} \sin(x)\,dx$

B) $\int_{\pi}^{-\pi} \sin(x)$ D) $\int_{0}^{\pi} \sin(x)\,dx$ F) $\int_{\pi}^{0} \sin(x)\,dx$

A. $y = \sin(x)$ 

A) 0 C) ⊖ E) ⊕ $C = F < A = B < D = E$

B) 0 D) ⊕ F) ⊖

Properties of the definite integral:

1. k = a constant, $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
2. if $f(x) < g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$
3. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
4. $\int_b^a (f(x) + g(x))\,dx = \int_b^a f(x)\,dx + \int_b^a g(x)\,dx$
5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
6. a) $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd

 b) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is even

Q. Choose the correct answer

$\int_{-a}^{a} f(x)\, dx$

(i) 0

(ii) $2\int_{0}^{a} f(x)\, dx$

(iii) $k\int_{-a}^{a} g(x)\, dx$

(iv) $\int_{-a}^{a} g(x)\, dx - \int_{-a}^{a} h(x)\, dx$

(a) $f(x) = g(x) - h(x)$

(b) $f(x) = kg(x)$

(c) $f(x) = \cos(x)$

(d) $f(x) = \sin(x)$

A. (a) iv (b) iii (c) ii (d) i

These are examples of the use of properties of definite integrals