Curves and Implicit Differentiation

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1 Lemniscate of Bernoulli

The *lemniscate of Bernoulli* is a curve defined by the equation

$$(x^2 + y^2)^2 = x^2 - y^2. \tag{1}$$

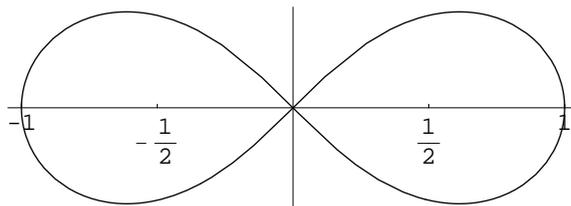The graph of this curve is a figure eight (Figure 1).

Figure 1: Lemniscate of Bernoulli

Suppose that we wish to find the x-coordinates of points on the curve that have a horizontal tangent line. To do this, we will use implicit differentiation and differentiate both sides of equation (1) with respect to x to obtain

$$2(x^2 + y^2)(2x + 2yy') = 2x - 2yy'.$$

This equation is equivalent to

$$2x^3 + 2xy^2 + 2x^2yy' + 2y^3y' = x - yy'.$$

Solving for y', we obtain

$$y' = \frac{x(1 - 2x^2 - 2y^2)}{y(1 + 2x^2 + 2y^2)}. \tag{2}$$

We know that we have a horizontal tangent line if $y' = 0$, and this will occur exactly when the numerator of the right-hand side of equation (2) is zero. That is, when

$$x(1 - 2x^2 - 2y^2) = 0.$$

We have two cases to consider, $x = 0$ and $1 - 2x^2 - 2y^2 = 0$.

If $x = 0$, then equation (1) becomes $y^4 = -y^2$. This is only possible if $y = 0$. However, if $y = 0$, then the derivative (2) is undefined since y occurs as a factor in the denominator.

The other possible case we must consider is $1 - 2x^2 - 2y^2 = 0$. If we solve this expression for y^2, then

$$y^2 = \frac{1 - 2x^2}{2}.$$

We can substitute the right-hand side of this equation for y^2 in equation (1) to get

$$\left(x^2 + \frac{1 - 2x^2}{2}\right)^2 = x^2 - \frac{1 - 2x^2}{2}.$$

Simplifying both sides of this equation, we get

$$\frac{1}{4} = \frac{4x^2 - 1}{2}.$$

We can solve this equation for x^2 to get $x^2 = 3/8$. Thus, we have horizontal tangent lines at $x = \pm\sqrt{3/8}$.

2   Folium of Descartes

As a second example of finding tangent lines to a curve using implicit differentiation, we can consider the *folium of Descartes*, which is defined by the equation

$$x^3 + y^3 = 6xy. \tag{3}$$

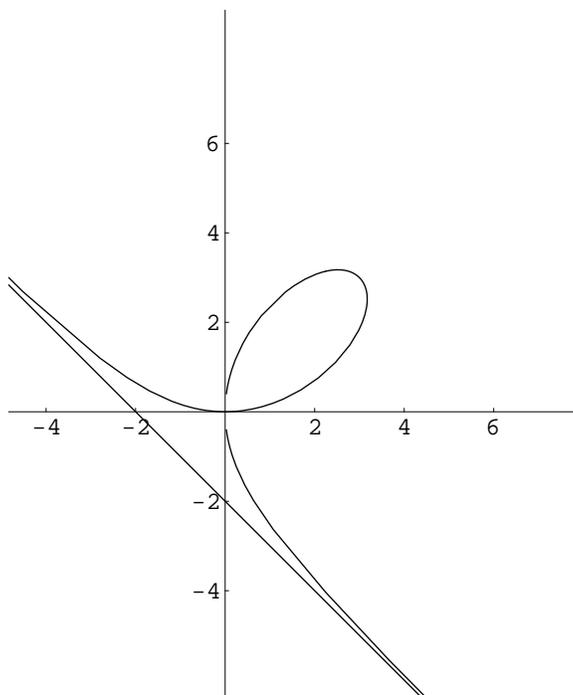The graph of this curve is shown in Figure 2.

Figure 2: Folium of Descartes

First, let us compute y'. Differentiating both sides of equation (3), we have

$$3x^2 + 3y^2 y' = 6(y + xy').$$

If we solve this equation for y', we obtain

$$y' = \frac{2y - x^2}{y^2 - 2x}.$$

It is easy to check that the point $(3, 3)$ is on the curve. In order to find an equation of the line that is tangent to the curve at this point, we must compute the derivative at this point,

$$y'|_{(3,3)} = \frac{2(3) - 3^2}{3^2 - 2(3)} = -1.$$

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Since we know a point on the line and the slope of the line, it is now easy to find an equation of the line,

$$y - 3 = -1(x - 3) \text{ or } y = -x + 6.$$

Finally, let us determine the points on the curve that have a horizontal tangent line. For a point to have a horizontal tangent line, we must have

$$y' = \frac{2y - x^2}{y^2 - 2x} = 0.$$

This will occur when the numerator of the fraction on the right-hand side of the equation is zero,

$$2y - x^2 = 0 \text{ or } y = \frac{x^2}{2}.$$

Substituting $y = x^2/2$ into equation (3), we have

$$x^3 + \left(\frac{x^2}{2}\right)^3 = 6x \left(\frac{x^2}{2}\right)$$

or if we simplify,

$$x^6 = 16x^3.$$

Consequently, $x^6 - 16x^3 = x^3(x^3 - 16) = 0$, and either $x = 0$ or $x^3 = 16$. Thus, we have horizontal tangent lines when $x = 0$ or $x = \sqrt[3]{16}$.