Chapter 17

Implicit Differentiation and its Applications

17.1 Introductory Example

1. (a) $y = 3^x \Rightarrow y' = (\ln 3)3^x$ by formula or think of $y = 3^x = e^{x(\ln 3)}$

(b) $y = x^3 \Rightarrow y' = 3x^2$

(c) $y = x^x = e^{x(\ln x)} \Rightarrow y' = e^{x(\ln x)}[(1)\ln x + x\frac{1}{x}] = x^x[\ln x + 1]$

2. $y = (x+1)^{(x+1)} = e^{(x+1)\ln(x+1)}$
$\Rightarrow y' = e^{(x+1)\ln(x+1)}[(1)\ln(x+1) + (x+1)\frac{1}{x+1}] = (x+1)^{(x+1)}[\ln(x+1)+1]$

3. $y = (3x^2+2)^x = (e^{\ln(3x^2+2)})^x = e^{x\ln(3x^2+2)}$
$\Rightarrow y' = e^{x\ln(3x^2+2)}[(1)\ln(3x^2+2) + x\frac{6x}{3x^2+2}] = (3x^2+2)^x[\ln(3x^2+2) + \frac{6x^2}{3x^2+2}]$

17.2

2. Sometimes we use logarithmic differentiation for the whole expression other times for subparts.

(a) $f(x) = 3\cdot 2^x + 2\cdot x^3 + 3\cdot x^{2x+3} = 3\cdot 2^x + 2\cdot x^3 + 3k(x)$ where
$\ln(k(x)) = \ln(x^{2x+3}) = (2x+3)\ln x \Rightarrow \frac{k'(x)}{k(x)} = 2\ln x + (2x+3)\frac{1}{x} \Rightarrow k'(x) = x^{2x+3}[2\ln x + \frac{2x+3}{x}]$
hence $f'(x) = 3(\ln 2)2^x + 6x^2 + 3\cdot x^{2x+3}[2\ln x + \frac{2x+3}{x}]$

(b) $f(x) = x(2x^3+1)^x + 5 = h(x) + 5 \Rightarrow f'(x) = h'(x) + 0$.
$\ln(h(x)) = \ln(x(2x^3+1)^x) = \ln x + x\ln(2x^3+1)$

$\Rightarrow \frac{h'(x)}{h(x)} = \frac{1}{x} + (1)\ln(2x^3+1) + x\frac{6x^2}{2x^3+1} = \frac{1}{x} + \ln(2x^3+1) + \frac{6x^3}{2x^3+1}$
$\Rightarrow h'(x) = x(2x^3+1)^x[\frac{1}{x} + \ln(2x^3+1) + \frac{6x^3}{2x^3+1}] = h'(x) + 0 = f'(x)$

3. (a) $\ln(g(t)) = \ln(\frac{2^t}{t^{2t}}) = t\ln 2 - 2t\ln t \Rightarrow \frac{g'(t)}{g(t)} = \ln 2 - [2\ln t + 2t\frac{1}{t}] \Rightarrow g'(t) = \frac{2^t}{t^{2t}}[\ln 2 - 2\ln t - 2]$

(b) $g(t) = \ln(t+1)^{t^2+1} = (t^2+1)\ln(t+1) \Rightarrow g'(t) = 2t\ln(t+1) + (t^2+1)\frac{1}{t+1} = 2t\ln(t+1) + \frac{t^2+1}{t+1}$

4. (a) $\ln y = \ln(x^{\ln\sqrt{x}}) = \ln x^{\frac{1}{2}\ln x} = \frac{1}{2}\ln x \ln x = \frac{1}{2}(\ln x)^2$
$\Rightarrow \frac{y'}{y} = \frac{1}{2}(2\ln x)\frac{1}{x} = \frac{\ln x}{x} \Rightarrow y' = \frac{\ln x}{x}x^{\ln\sqrt{x}} = (\ln x)x^{\ln\sqrt{x}-1}$

(b) $\ln y = \ln\left(\frac{xe^{5x}}{(x+1)^2\sqrt{x-2}}\right) = \ln x + \ln e^{5x} - \ln(x+1)^2 - \ln\sqrt{x-2} = \ln x + 5x - 2\ln(x+1) - \frac{1}{2}\ln(x-2)$
$\Rightarrow \frac{y'}{y} = \frac{1}{x} + 5 - 2\frac{1}{x+1} - \frac{1}{2}\frac{1}{x-2} \Rightarrow y' = (\frac{1}{x} + 5 - \frac{2}{x+1} - \frac{1}{2x-4})\left(\frac{xe^{5x}}{(x+1)^2\sqrt{x-2}}\right)$

(c) $\ln y = \ln(e^{2x}(x^2+3)^5(2x^2+1)^3) = \ln e^{2x} + \ln(x^2+3)^5 + \ln(2x^2+1)^3$
$= 2x + 5\ln(x^2+3) + 3\ln(2x^2+1) \Rightarrow \frac{y'}{y} = 2 + 5\frac{2x}{x^2+3} + 3\frac{4x}{2x^2+1}$
$\Rightarrow y' = (2 + \frac{10x}{x^2+3} + \frac{12x}{2x^2+1})(e^{2x}(x^2+3)^5(2x^2+1)^3)$

7. $y = f(x)^{g(x)} \Rightarrow \ln y = \ln\left(f(x)^{g(x)}\right) = g(x)\ln(f(x)) \Rightarrow \frac{y'}{y} = g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}$
$\Rightarrow y' = \left(g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right)f(x)^{g(x)}$