17.3 Implicit Differentiation

1\. (a) $x^2 + y^2 = 169 \Rightarrow y^2 = 169 - x^2 \Rightarrow y = \pm(169 - x^2)^{1/2}$. This is not a function.

(b) $$\frac{dy}{dx} = \begin{cases} \frac{1}{2}(169 - x^2)^{-1/2}(-2x) = -x(169 - x^2)^{-1/2} = \frac{-x}{(169-x^2)^{1/2}} & \text{if } y = (169 - x^2)^{1/2} \\ -\frac{1}{2}(169 - x^2)^{-1/2}(-2x) = x(169 - x^2)^{-1/2} = \frac{x}{(169-x^2)^{1/2}} & \text{if } y = -(169 - x^2)^{1/2} \end{cases}$$

(c) $2x + 2y\frac{dy}{dx} = 0 \Rightarrow 2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

(d) Method (c) is easier. To find $\frac{dy}{dx}$ in terms of x substitute for y in each of the two cases.

(e) $\frac{dy}{dx}\Big|_{(5,12)} = \frac{-5}{12}$ and $\frac{dy}{dx}\Big|_{(5,-12)} = \frac{-5}{-12} = \frac{5}{12}$

3\. $x^2y + xy^2 + x = 1 \Rightarrow (x)y^2 + (x^2)y + (x-1) = 0$ is of form $ay^2 + by + c = 0$. To find all points at which $x = 1$ use the quadratic formula on $(1)^2y + (1)y^2 + (1) = 1$ or $y^2 + y = 0 \Rightarrow y(y+1) = 0$ $\Rightarrow y = 0, -1$. To find the slope use implicit differentiation to get $2xy + x^2\frac{dy}{dx} + y^2 + x2y\frac{dy}{dx} + 1 = 0$. At (1,0), $0 + 1\frac{dy}{dx} + 0 + 0 + 1 = 0 \Rightarrow \frac{dy}{dx} = -1$.
At (1,−1), $-2 + \frac{dy}{dx} + 1 - 2\frac{dy}{dx} + 1 = 0 \Rightarrow -\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = 0$.

6\. $x^3 + 3y + y^2 = 6 \Rightarrow 3x^2 + 3\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \Rightarrow (3 + 2y)\frac{dy}{dx} = -3x^2 \Rightarrow \frac{dy}{dx} = \frac{-3x^2}{3+2y}$.
Hence $\frac{dy}{dx}\Big|_{(2,-1)} = \frac{-3(4)}{3+2(-1)} = -12$.
$\frac{dy}{dx} = 0$ when $x = 0 \Rightarrow 0 + 3y + y^2 = 6 \Rightarrow y^2 + 3y - 6 = 0 \Rightarrow y = \frac{-3\pm\sqrt{9-4(6)}}{2} = \frac{-3\pm\sqrt{33}}{2}$.
Hence the slope = 0 at points $(0, \frac{-3+\sqrt{33}}{2})$ and $(0, \frac{-3-\sqrt{33}}{2})$.

12\.(a) $6x + 12y\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} = 0 \Rightarrow (12y + 3x)\frac{dy}{dx} = -6x - 3y \Rightarrow \frac{dy}{dx} = \frac{-(2x+y)}{(x+4y)}$

(b) $(x-2)^3 = (y-2)^{-3} \Rightarrow 3(x-2)^2 = -3(y-2)^{-4}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -(x-2)^2(y-2)^4 = -\frac{(y-2)}{(x-2)}$

(c) $y^2 + x(2y\frac{dy}{dx}) + 2\frac{dy}{dx} = 2xy + x^2\frac{dy}{dx} \Rightarrow (2xy + 2 - x^2)\frac{dy}{dx} = 2xy - y^2 \Rightarrow \frac{dy}{dx} = \frac{2xy-y^2}{2xy+2-x^2}$

(d) $2(x^2y^3 + y)(2xy^3 + x^2(3y^2\frac{dy}{dx}) + \frac{dy}{dx}) = 3 \Rightarrow 2xy^3 + (3x^2y^2 + 1)\frac{dy}{dx} = \frac{3}{2(x^2y^3+y)}$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{3}{2(x^2y^3+y)} - 2xy^3}{3x^2y^2 + 1} = \frac{3 - 4x^3y^6 - 4xy^4}{2(x^2y^3 + y)(3x^2y^2 + 1)} = \frac{3 - 4x^3y^6 - 4xy^4}{6x^4y^5 + 8x^2y^3 + 2y}$$

(e) $e^{xy}(y + x\frac{dy}{dx}) = 2y\frac{dy}{dx} \Rightarrow xe^{xy}\frac{dy}{dx} - 2y\frac{dy}{dx} = -ye^{xy} \Rightarrow \frac{dy}{dx} = \frac{ye^{xy}}{2y-xe^{xy}}$

(f) $x(\ln x + 3\ln y) = y^2 \Rightarrow x\ln x + 3x\ln y = y^2 \Rightarrow$
$\ln x + x\frac{1}{x} + 3\ln y + 3x\frac{1}{y}\frac{dy}{dx} = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{\ln x+1+3\ln y}{2y-\frac{3x}{y}} = \frac{y(1+\ln x+3\ln y)}{2y^2-3x}$

(g) $\frac{1}{xy}(y + x\frac{dy}{dx}) = y^2 + 2xy\frac{dy}{dx} \Rightarrow \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = y^2 + 2xy\frac{dy}{dx} \Rightarrow (\frac{1}{y} - 2xy)\frac{dy}{dx} = y^2 - \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{xy^3-y}{x-2x^2y^2}$

14\. $x^{2/3} + y^{2/3} = 5 \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-y^{1/3}}{x^{1/3}} \Rightarrow \frac{dy}{dx}\Big|_{(8,1)} = \frac{-1}{2}$. The line is $y = -\frac{1}{2}x + 5$.