Math Xb Spring 2004

Lab 8: Appreciating the FTC

April 20, 2004

Name SOLUTIONS

Section Instructor ________

Collaborators ________

Before our discussion of the Fundamental Theorem of Calculus, we did not have a convenient way of evaluating an integral like

$$\int_0^2 x^3\,dx.$$

Since the area under the function $f(x) = x^3$ is not a geometric area with which we are familiar, the best we could do is approximate the integral using left- or right-hand sums.

However, it is possible to evaluate such an integral even without the Fundamental Theorem of Calculus. In the following exercises, you will evaluate the integral given above *without* using the FTC. The purpose of these exercises is to remind you of the limit definition of the definite integral and to help you appreciate the usefulness of the FTC.

1. To evaluate $\int_0^2 x^3\,dx$, first partition the interval $[0, 2]$ into n subintervals. Each subinterval then has width equal to Δx. Give a formula for Δx in which n is the only variable.

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \boxed{\frac{2}{n}}$$

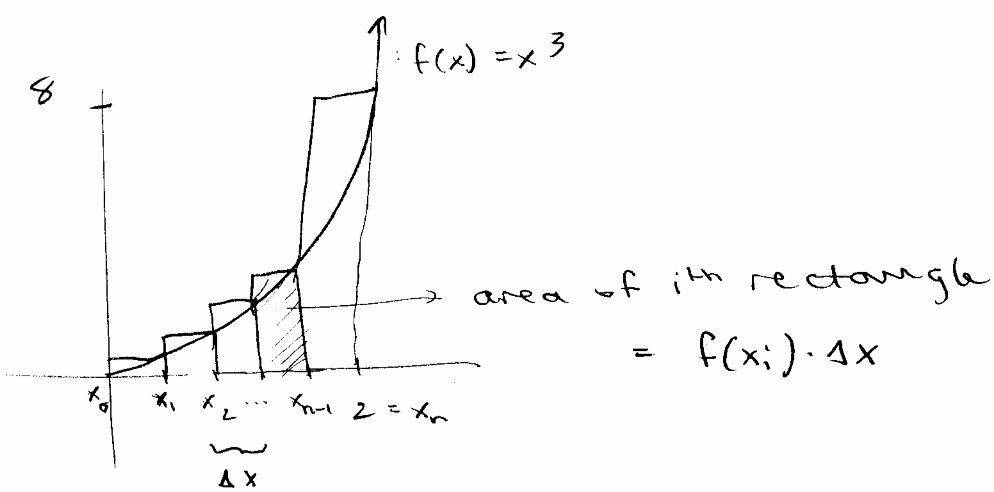
2. Label the endpoints of the subintervals $x_0, x_1, \ldots, x_n$. Give a formula for x_i in which i and n are the only variables.

$$x_i = a + i \cdot \Delta x = 0 + i\left(\frac{2}{n}\right) = \boxed{\frac{2i}{n}}$$

3. Recall that the right-hand sum approximation to the definite integral using n subintervals is given by

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

Draw a picture illustrating this formula.



4. Express R_n for the integral $\int_0^2 x^3\,dx$ as a sum in which i and n are the only variables. (You will need to use your answers to questions 1 and 2 as well as the formula given in question 3.)

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\cdot\frac{2}{n}$$

$$\boxed{= \sum_{i=1}^{n}\left(\frac{2i}{n}\right)^3\cdot\frac{2}{n}}$$

5. Simplify the expression you found for R_n in question 4 using the fact that

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Your answer should be an expression in which n is the only variable.

$$R_n = \sum_{i=1}^{n}\left(\frac{2i}{n}\right)^3\cdot\frac{2}{n} = \sum_{i=1}^{n}\left(\frac{2}{n}\right)^3\cdot i^3\cdot\frac{2}{n}$$

$$= \left(\frac{2}{n}\right)^4\cdot\sum_{i=1}^{n} i^3 = \boxed{\left(\frac{2}{n}\right)^4\cdot\left[\frac{n(n+1)}{2}\right]^2}$$

6. Recall that

$$\int_0^2 x^3\,dx = \lim_{n\to\infty} R_n.$$

Find $\lim_{n\to\infty} R_n$ using the expression you found in the question 5. Your answer should be a number.

$$\int_0^2 x^3\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(\frac{2}{n}\right)^4\left[\frac{n(n+1)}{2}\right]^2$$

$$= \lim_{n\to\infty}\frac{2^4\cdot n^2(n+1)^2}{n^4\,2^2} = \lim_{n\to\infty}\frac{2^2(n+1)^2}{n^2}$$

$$= \lim_{n\to\infty}\frac{4n^2+8n+4}{n^2} = \boxed{4}$$

7. Now evaluate $\int_0^2 x^3\,dx$ using the Fundamental Theorem of Calculus (Version 2). Does your answer agree with the value you found in question 6?

$$\int_0^2 x^3\,dx = \frac{1}{4}x^4\Big|_0^2 = \frac{1}{4}2^4 - \frac{1}{4}0^4 = \boxed{4}$$ ✓