Worksheet : The Substitution Rule (Day 2) SOLUTIONS

4/23/04

① $\int x^5 \sqrt{1+x^2}\, dx = \int x^4 \underbrace{\sqrt{1+x^2}}_{u}\ \underbrace{x\, dx}_{\frac{1}{2}du} = \int (u-1)^2 \sqrt{u} \cdot \frac{1}{2}\, du$

$u = 1+x^2$

$du = 2x\, dx \longrightarrow x^2 = u-1$

$\frac{1}{2} du = x\, dx \qquad x^4 = (u-1)^2$

$= \frac{1}{2}\int (u^2 - 2u + 1)\, u^{1/2}\, du = \frac{1}{2}\int \left(u^{5/2} - 2u^{3/2} + u^{1/2} \right) du$

$= \frac{1}{2}\left(\frac{2}{7} u^{7/2} - 2 \cdot \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right) + C$

$= \frac{1}{7} u^{7/2} - \frac{2}{5} u^{5/2} + \frac{1}{3} u^{3/2} + C$

$= \boxed{\frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C}$

② $\frac{4}{7}(x+2)^{7/4} - \frac{8}{3}(x+2)^{3/4} + C$

③ $\int \frac{1+x}{1+x^2}\, dx = \int \left(\frac{1}{1+x^2} + \frac{x}{1+x^2} \right) dx = \int \frac{1}{1+x^2}\, dx + \underbrace{\int \frac{x}{1+x^2}\, dx}$

$u = 1+x^2$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$= \tan^{-1} x + \frac{1}{2} \int \frac{1}{u}\, du$

$= \tan^{-1} x + \frac{1}{2} \ln |u| + C$

$= \boxed{\tan^{-1} x + \frac{1}{2} \ln (1+x^2) + C}$

④ $-2(1-x)^{1/2} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2} + C$

⑤ $\displaystyle\int \frac{25}{25+9x^2}\,dx = \int \frac{25}{25+9x^2}\cdot\frac{1/25}{1/25}\,dx = \int \frac{1}{1+\frac{9x^2}{25}}\,dx$

$\displaystyle = \int \frac{1}{1+\left(\frac{3x}{5}\right)^2}\,dx = \frac{5}{3}\int \frac{1}{1+u^2}\,du = \frac{5}{3}\tan^{-1}u + C$

$\displaystyle = \boxed{\frac{5}{3}\tan^{-1}\left(\frac{3}{5}x\right) + C}$

$u = \frac{3}{5}x$

$du = \frac{3}{5}\,dx$

$\frac{5}{3}\,du = dx$

⑥ $\frac{1}{7}(x^2+1)^{7/2} - \frac{1}{5}(x^2+1)^{5/2} + C$

⑦ $\frac{10}{3}$

⑧ $\displaystyle\int_0^1 x\sqrt{1-x^4}\,dx = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du$

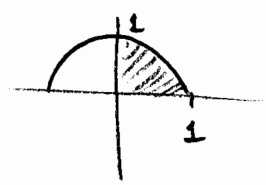
$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$y = \sqrt{1-u^2}$ is the top half of the circle centered at the origin with radius 1.

(Why? $y = \sqrt{1-u^2}$
$y^2 = 1-u^2$
$u^2+y^2 = 1$.)



$\Rightarrow$

$\displaystyle\frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du$

$\displaystyle = \frac{1}{2}\cdot\frac{1}{4}\cdot\pi(1)^2 = \boxed{\frac{\pi}{8}}$

⑨ volume at time x $\displaystyle = \int_0^x \frac{1}{2}\sin\left(\frac{2\pi t}{5}\right)dt$

$u = \frac{2\pi t}{5}$

$du = \frac{2\pi}{5}\,dt$

$\displaystyle = \frac{1}{2}\cdot\frac{5}{2\pi}\int_0^{\frac{2\pi x}{5}} \sin u\,du$

$\displaystyle = \frac{-5}{4\pi}\cos\Big|_0^{\frac{2\pi x}{5}} = \frac{-5}{4\pi}\left(\cos\frac{2\pi x}{5} - 1\right)$

$\displaystyle \Rightarrow V(t) = \boxed{\frac{-5}{4\pi}\left(\cos\frac{2\pi t}{5} - 1\right)}$