Math Xb Spring 2004
Worksheet: Logarithmic Differentiation
February 11, 2004

1. Let $f(x) = x^x$.

   (a) Use the definition of the derivative to approximate $f'(2)$ numerically.

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^{(2+h)} - 4}{h}$$

$$\approx \frac{(2+.001)^{(2+.001)} - 4}{.001} \approx 6.78$$

   (b) Attempt to find $f'(x)$.

$\frac{d}{dx}(x^n) = nx^{n-1}$ doesn't apply!

$\frac{d}{dx}(b^x) = b^x \ln b$ doesn't apply!

   (c) Use your answer to part (b) to find $f'(2)$ and compare this with your answer to part (a).

2. Find the derivative of each of the following functions.

   (a) $f(x) = x^{\ln x}$

$$\ln(f(x)) = \ln x^{\ln x}$$

$$\ln(f(x)) = \ln x \cdot \ln x$$

$$\frac{f'(x)}{f(x)} = \ln x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{x}$$

$$f'(x) = f(x)\left(\frac{2\ln x}{x}\right)$$

$$= \boxed{x^{\ln x}\left(\frac{2\ln x}{x}\right)}$$

   (b) $f(x) = (\ln x)^x$

$$\ln(f(x)) = \ln(\ln x)^x$$

$$\ln(f(x)) = x\ln(\ln x)$$

$$\frac{f'(x)}{f(x)} = x\frac{d}{dx}(\ln(\ln x)) + \ln(\ln x)\cdot\frac{d}{dx}(x)$$

$$f'(x) = (\ln x)^x\left(x \cdot \frac{1}{\ln x}\cdot x + \ln(\ln x)\right)$$

(c) $f(x) = 3^x + x^3 + x^{3x}$

$$f'(x) = 3^x \ln 3 + 3x^2 + x^{3x}(3 + 3\ln x)$$

$$g(x) = x^{3x}$$

$$\ln(g(x)) = 3x \cdot \ln x$$

$$\frac{g'(x)}{g(x)} = 3x \cdot \frac{1}{x} + 3 \cdot \ln x$$

$$g'(x) = x^{3x}(3 + 3\ln x)$$

3. Use logarithmic differentiation to find the derivative of $f(x) = \frac{xe^x}{(x^2+2)^4(5x+2)^2}$.

$$\ln(f(x)) = \ln\left(\frac{xe^x}{(x^2+2)^4(5x+2)^2}\right)$$

$$= \ln x + \ln e^x - \ln(x^2+2)^4 - \ln(5x+2)^2$$

$$= \ln x + x - 4\ln(x^2+2) - 2\ln(5x+2)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{1}{x} + 1 - 4 \cdot \frac{1}{x^2+2} \cdot 2x - 2 \cdot \frac{1}{5x+2} \cdot 5$$

$$f'(x) = \frac{xe^x}{(x^2+4)^4(5x+2)^2}\left(\frac{1}{x} + 1 - \frac{8x}{x^2+2} - \frac{10}{5x+2}\right)$$

4. Suppose $y = f(x)g(x)$, where $f(x)$ and $g(x)$ are positive for all x.

(a) Use logarithmic differentiation to find $\frac{dy}{dx}$.

$$\ln y = \ln(f(x)g(x))$$

$$\ln y = \ln f(x) + \ln g(x)$$

$$y'/y = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}$$

$$y' = y\left(\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right) = f(x)g(x)\left(\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right)$$

$$= g(x)f'(x) + f(x)g'(x)$$

(b) Use the Product Rule to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$$

(c) Do your answers to parts (a) and (b) agree?

Yes!