Math Xb Spring 2005
Sine and Cosine Functions

1 Goals

- Understand how to define the sine and cosine functions as a function of arc length on the unit circle
- Be able to approximate the sine and cosine using a calibrated unit circle
- Be familiar with the graphs of the sine and cosine functions
- To understand certain properties of the sine and cosine functions, such as periodicity, odd/even, and some trigonometric identities

2 New Terms

- Arc Length
- Unit Circle
- Sine, Cosine
- Periodic function

3 Introduction

1. *Problem:* The gravitational pull of the sun and moon on large bodies of water produced tides. Tides generally rise and fall twice every 25 hours. The range between high and low tide varies greatly with location. On the Pacific coast of North America this range can be as much as 15 feet. The Bay of Fundy in New Brunswick has a dramatic range of 45 feet.

2. Here's a tidal record for two weeks in 1887 at Bombay, showing the height of the tides each day as function of time (note that time increases to the left, not the right). Each curve represents a day.

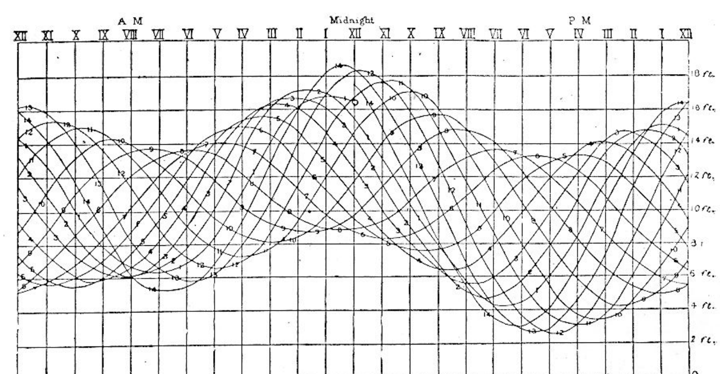
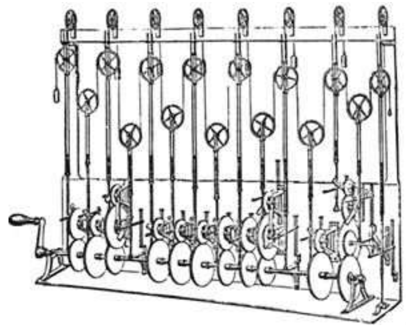


FIG. 3.—Tide-curve for Bombay from the beginning of the civil year 1884 to the midnight ending Jan. 14, 1884, or from 12h Dec. 31, 1883, to 12h Jan. 14, 1884, astronomical time.

3. Can we predict when high tide and low tide can occur? Lord Kelvin predicted tides using a machine like the one below. We'd like to be able to come up with a mathematical function that we could use to model the height of the tides.



4. Note that the functions we have seen so far (polynomial, rational, exponential, and logarithmic functions) are not good models for tidal behavior. We need a function that is period, that repeats itself over and over again. That's going to be the subject of our next unit of material, trigonometry.

4 Definition of Sine and Cosine

1. The *unit circle* is a circle of radius 1 centered at the origin.
2. Label the horizontal axis with a u and the vertical axis with a v.
3. Note that any point P on the unit circle can located by stating how far a distance x to travel along the along the circle to get to the point. Here we are taking the input, x to be the *arc length* from $(1, 0)$ to the point P.
   (a) Start at $(1, 0)$.
   (b) If $x \geq 0$, travel along the circle in a counter-clockwise direction by x units to arrive at $P(x)$.
   (c) If $x < 0$, travel along the circle in a clockwise direction by x units to arrive at $P(x)$.
4. But remember that we can also locate a point on the unit circle by giving a u and v coordinate. Thus we can make the following definitions.
5. Define $\cos x$ to be the u-coordinate of $P(x)$, that is, $\cos x$ is the first coordinate of the point $P(x)$.
6. Define $\sin x$ to be the v-coordinate of $P(x)$, that is, $\sin x$ is the second coordinate of the point $P(x)$.

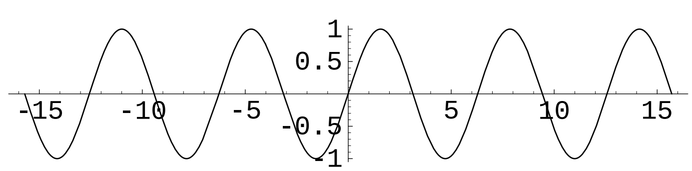
5 Graphs of the Sine and Cosine (and periodicity)

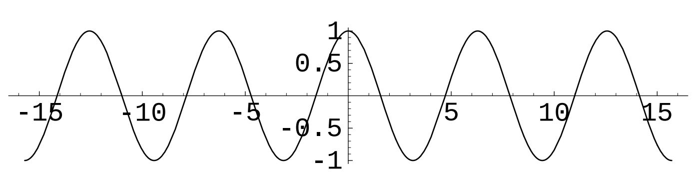
A function $f(x)$ is called a *periodic function* if there is a positive constant k such that $f(x + k) = f(x)$ for all x in the domain. In other words, a periodic function repeats itself at regular intervals. You may want to sketch up an example or two. The book has examples on page 596.

The sine and cosine are both periodic functions. Students have seen the sine and cosine previously, so they should be able to describe the shape of the graphs. Take some time to connect the shape of the graphs to the function values students have found from the unit circle.

Note that the period of both the sine and cosine functions is 2π. This means that

$$\begin{aligned} \sin(x + 2\pi) &= \sin(x) \\ \sin(x + 4\pi) &= \sin(x) \\ \sin(x + 2\pi n) &= \sin(x) \quad n \text{ any integer} \\ \cos(x + 2\pi n) &= \cos(x) \quad n \text{ any integer} \end{aligned}$$





Also notice that the sine and cosine functions are slightly "out of phase" with each other. If we shift the cosine graph a little to the right, we get the graph of the sine. The shift needed is exactly $\frac{\pi}{2}$. This gives us the following relationship

$$\sin(x) = \cos\left(x - \tfrac{\pi}{2}\right).$$

6 Properties of the Sine and Cosine

1. The sine is an odd function $\sin(-x) = -\sin x$. We can see this by using the unit circle: for x a positive number, $\sin x$ is the y-coordinate of the point we get to by traveling x units counter-clockwise, while $\sin -x$ is the y-coordinate of the point we get to by traveling x units clockwise. The cosine is an even function and $\cos(-x) = \cos x$ (similar reasoning).

2. Since $(\cos x, \sin x)$ is a point on the unit circle, $\sin^2 x + \cos^2 x = 1$. Note that $\sin^2 x = (\sin x)^2$, and neither of these is equal to $\sin(x^2)$.

7 References

Section 19.1 in *Calculus: An Integrated Approach to Functions and Their Rates of Change.*