Math Xb Spring 2005
L'Hôpital's Rule Day One

March 18, 2005

1 Goals

- To recognize the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.
- To understand L'Hôpital's Rule in terms of relative rates of change.
- To evaluate the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ using L'Hôpital's Rule.

2 New Terms

- Indeterminate forms
- L'Hôpital's Rule

3 Indeterminate Forms

1. Consider the following examples
   (a) $\lim_{x\to\infty} \frac{2x^3-x+1}{x^3+3x}$
   (b) $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$
2. In the first example, both the numerator and denominator approach ∞. Since the numerator grows without bound, we might expect the limit to be ∞. Since the denominator grows without bound, we might expect the limit to be 0. However, in some sense, the numerator approaches ∞ *twice as fast* as the denominator, and so the limit is 2.
3. In the second example, both the numerator and denominator approach 0. Since the numerator approaches 0, we might expect the limit to be 0. Since the denominator approaches 0, we might expect the limit to be ∞. In fact, we know that this limit equals $f'(a)$, which could be anything!
4. We call the first limit an *indeterminate form* of type $\frac{\infty}{\infty}$.
5. We call the second limit an indeterminate form of type $\frac{0}{0}$.
6. **Definition:** If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then we call the expression
   $$\lim_{x\to a} \frac{f(x)}{g(x)}$$
   an *indeterminate form* of type $\frac{0}{0}$.
7. **Definition:** If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$, then we call the expression
   $$\lim_{x\to a} \frac{f(x)}{g(x)}$$
   an *indeterminate form* of type $\frac{\infty}{\infty}$.

4 L'Hôpital's Rule

1. **L'Hôpital's Rule:** If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$
This result also holds if a is replaced by ∞, $-\infty$, a^+, or a^-.

2. Note that L'Hôpital's Rule doesn't say to take the limit of the derivative of the *quotient* but rather to take the derivatives of the numerator and denominator individually, and then take the limit.

3. Note that we can only apply L'Hôpital's Rule if we meet all of the conditions. We must have the a limit in one of the two indeterminate forms, and we must have both derivatives existing and the limit existing.

4. Why does L'Hôpital's Rule work? This is described on page 1114–1115 of the textbook, but only for the special case of
$$\lim x \to a \frac{f(x)}{g(x)}$$
Where $f(a) = g(a) = 0$ and $g'(a) \neq 0$. The heuristic idea is that we first use a linear approximation at $x = a$ to approximate $f(x)$ and $g(x)$.
$$\begin{aligned}\lim_{x\to a} \frac{f(x)}{g(x)} &\approx \lim_{x\to a} \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)} \\ &\approx \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} \\ &\approx \frac{f'(a)}{g'(a)}\end{aligned}$$
A proof in this special case can be found in the book on p. 1115.

5 References

- Appendix F in *Calculus: An Integrated Approach to Functions and Their Rates of Change.*