Math Xb Spring 2005
Net Change

March 23, 2005

1 Goals

- To calculate the area under certain simple curves.
- To approximate the area under a curve using left- and right-hand sums.
- To understand the definition of the definite integral.
- To understand the notion of signed area under a curve.
- To calculate certain simple definite integrals.

2 New Terms

- Definite integral
- Integrand
- Endpoints (or limits) of integration
- Subinterval

3 Net Change and Area Under a Curve

1. *Problem:* Suppose that a car's velocity (in meters per second) is given by $v(t) = 3t - 2$ on the time interval $[2, 4]$. What is the net change in the car's position between $t = 2$ and $t = 4$?
2. We saw in the previous lesson that we can approximate the net change in the car's position using left- and right-hand sums. As the number of intervals increase, the approximations get better *and* converge to the area under the curve.
3. Thus, the net change in position is equal to the area under the curve $v(t) = 3t - 2$ between $t = 2$ and $t = 4$. Since that area is a trapezoid, we can compute the area, and thus the net change, exactly.
4. Suppose that the car's velocity is given by $v(t) = 4t - t^2$. Can we find the net change in position between $t = 2$ and $t = 4$?
5. We can still answer this question in terms of area under a curve, but now the area is not a nice geometric figure. We cannot find the exact area (yet), but we can still approximate it with left- and right-hand sums.
6. Note that as the number of intervals increases, the approximations get better *and* converge to the area under the curve.

4 The Definite Integral

1. The *definite integral* is a mathematical tool that helps us answer the following questions.

   (a) Given a rate of change $f'(x)$, how do we calculate the net change in $f(x)$ over an interval?

   (b) How do we calculate the area under the graph of a function?

2. Partition the interval $[a, b]$ into n equal subintervals, each of width $\Delta x = (b-a)/n$. Let $x_i = a + i \cdot \Delta x$ for $i = 0, 1, \ldots, n$. Then the subintervals are $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$.

3. Note that $x_0 = a + 0 \cdot \Delta x = a$ and $x_n = a + n \cdot \Delta x = a + n \cdot (b-a)/n = a + (b-a) = b$.

4. We define the *left-hand sum* of f on $[a, b]$ with n subintervals to be

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x.$$

   Be sure to introduce the summation notation to them here and explain how it works. The notation is covered in section 18.4 beginning on p. 575 of the textbook.

5. We define the *right-hand sum* of f on $[a, b]$ with n subintervals to be

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$

6. We saw that as n gets large, both R_n and L_n converge to the same amount—the net change in amount or the area under the curve, depending on how you interpret them. What is the difference between R_n and L_n?

$$\begin{aligned}
|R_n - L_n| &= |(f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x) - \\
&\qquad (f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x))| \\
&= |f(x_n)\Delta x - f(x_0)\Delta x| \\
&= |f(x_n) - f(x_0)|\Delta x
\end{aligned}$$

   As n gets large, the above quantity gets closer to zero, Thus $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$, provided these limits exist.

7. The *definite integral* of f from $x = a$ to $x = b$ is written $\int_a^b f(x)\,dx$ and read as "the integral from a to b of $f(x)$." It is defined to be

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n,$$

   provided these limits exist.

8. *Fact:* If f is continuous on $[a, b]$, then these limits are guaranteed to exist.

9. The numbers a and b are called the *endpoints of integration* or *limits of integration.* The function $f(x)$ being integrated is called the *integrand.*

10. Show the notational relationship between $\int_a^b f(x)\,dx$ and $\sum_a^b f(x_i)\Delta x$.