Math Xb Spring 2005
Introduction to the Definite Integral

March 25, 2005

1 Goals

- To calculate the area under certain simple curves.
- To approximate the area under a curve using left- and right-hand sums.
- To understand the definition of the definite integral.
- To calculate certain simple definite integrals.

2 New Terms

- Definite integral
- Integrand
- Endpoints (or limits) of integration
- Subinterval

3 Net Change and Area Under a Curve

1. *Problem:* Suppose that a car's velocity (in meters per second) is given by $v(t) = 3t - 2$ on the time interval $[2, 4]$. What is the net change in the car's position between $t = 2$ and $t = 4$?
2. We saw in the previous lesson that we can approximate the net change in the car's position using left- and right-hand sums. As the number of intervals increase, the approximations get better *and* converge to the area under the curve.
3. Thus, the net change in position is equal to the area under the curve $v(t) = 3t - 2$ between $t = 2$ and $t = 4$. Since that area is a trapezoid, we can compute the area, and thus the net change, exactly.
4. Suppose that the car's velocity is given by $v(t) = 4t - t^2$. Can we find the net change in position between $t = 2$ and $t = 4$?
5. We can still answer this question in terms of area under a curve, but now the area is not a nice geometric figure. We cannot find the exact area (yet), but we can still approximate it with left- and right-hand sums.
6. Note that as the number of intervals increases, the approximations get better *and* converge to the area under the curve.

4 The Definite Integral

1. The *definite integral* is a mathematical tool that helps us answer the following questions.
   (a) Given a rate of change $f'(x)$, how do we calculate the net change in $f(x)$ over an interval?
   (b) How do we calculate the area under the graph of a function?

2. Partition the interval $[a, b]$ into n equal subintervals, each of width $\Delta x = (b - a)/n$. Let $x_i = a + i \cdot \Delta x$ for $i = 0, 1, \ldots, n$. Then the subintervals are $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$.

3. Note that $x_0 = a + 0 \cdot \Delta x = a$ and $x_n = a + n \cdot \Delta x = a + n \cdot (b - a)/n = a + (b - a) = b$.

4. We define the *left-hand sum* of f on $[a, b]$ with n subintervals to be

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x.$$

   The summation notation is covered in section 18.4 beginning on p. 575 of the textbook.

5. We define the *right-hand sum* of f on $[a, b]$ with n subintervals to be

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$

6. We saw that as n gets large, both R_n and L_n converge to the same amount—the net change in amount or the area under the curve, depending on how you interpret them. What is the difference between R_n and L_n?

$$\begin{aligned} |R_n - L_n| &= |(f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x) - \\ &\qquad (f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x))| \\ &= |f(x_n)\Delta x - f(x_0)\Delta x| \\ &= |f(x_n) - f(x_0)|\Delta x \end{aligned}$$

   As n gets large, the above quantity gets closer to zero, Thus $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$, provided these limits exist.

7. The *definite integral* of f from $x = a$ to $x = b$ is written $\int_a^b f(x)\,dx$ and read as "the integral from a to b of $f(x)$." It is defined to be

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n,$$

   provided these limits exist.

8. *Fact:* If f is continuous on $[a, b]$, then these limits are guaranteed to exist.

9. The numbers a and b are called the *endpoints of integration* or *limits of integration.* The function $f(x)$ being integrated is called the *integrand.*

10. Show the notational relationship between $\int_a^b f(x)\,dx$ and $\sum_a^b f(x_i)\Delta x$.