Math Xb Spring 2005
The Fundamental Theorem of Calculus, Version 2

April 13, 2005

1 Goals

- Understand the Fundamental Theorem of Calculus, part II
- Be able to use FTC, part II to calculate definite integrals, find area under curves, and determine net change given rate of change

2 New Terms

- Antiderivative, p. 761 in textbook.

3 Introductory Problem

1. Suppose a car's velocity for some period of time as it drives west of Houston, Texas, is given (in miles per hour) by the function $v(t) = 3t + 1$ on the time interval $[0, 10]$. Suppose we want to know the distance travelled by the car on that time interval, that is, we want to find
$$\int_0^{10} (3t + 1)\, dt.$$
2. One way would be to compute the area of the appropriate trapezoid.
3. Here is another way: We know that the car's velocity function is $v(t) = 3t + 1$, and we know that the velocity function is the derivative of the car's position function. The car's position function could thus be $p(t) = \frac{3}{2}t^2 + t$, $p(t) = \frac{3}{2}t^2 + t + 10$, $p(t) = \frac{3}{2}t^2 + t + 30$, etc.
4. We call each of these functions an *antiderivative* of v.
5. *Definition:* A function F is an *antiderivative* of f if $F' = f$.
6. Any antiderivative of $v(t) = 3t + 1$ must be equal to the function $p(t) = \frac{3}{2}t^2 + t$ plus some constant.
7. We can use this to compute the car's net change in position on the time interval $[0, 10]$. Note that the answer is the same no matter which position function is used.
8. What have we seen here? We have seen that
$$\int_0^{10} f(t)\, dt = p(10) - p(0),$$
where p is any antiderivative of v. This seems clear when $p(t)$ is a rate of change, but is it true in general?

4 FTC, Part 2

1. Recall that last time we proved the FTC, Version 1, which said that for a continuous function $f(t)$,

$$ {}_cA_f(x) = \int_c^x f(t)\,dt $$

is differentiable on (a, b) and

$$ \frac{d}{dx}\left({}_cA_f(x)\right) = f(x). $$

2. Using the notion of the antiderivative, we can can rephrase this as saying that $\int_c^x f(t)\,dt$ is an antiderivative of $f(x)$

3. Connecting with our example above, we see that

   (a) Any antiderivative of $v(t) = 3x + 1$ must be equal to the function $p(x) = \frac{3}{2}x^2 + x$ plus some constant, and

   (b) ${}_0A_f(x)$ is an antiderivative by the FTC, Part 1.

   Thus

$$ {}_0A_f(x) = \int_0^x f(t)\,dt = \frac{3}{2}x^2 + x + K $$

   for *some* constant K. (This is true some constant, not any constant.)

4. Since we don't know the constant C, this doesn't help us calculate a definite integral. But wait! Since $\int_c^x f(t)\,dt$ is an antiderivative of $f(x)$ we know that it differs from any *other* antiderivative, $F(x)$, by a constant

$$ F(x) = \int_c^x f(t)\,dt + C $$

   for some constant C. Note that this means

$$ F(a) = \int_c^a f(t)\,dt + C \quad \text{and} \quad F(b) = \int_c^b f(t)\,dt + C. $$

   Now we pull a cute trick using some of the properties of definite integrals to find $\int_a^b f(t)\,dt$.

$$ \begin{aligned} \int_a^b f(t)\,dt &= \int_a^c f(t)\,dt + \int_c^b f(t)\,dt \\ &= -\int_c^a f(t)\,dt + \int_c^b f(t)\,dt \\ &= \int_c^b f(t)\,dt - \int_a^c f(t)\,dt \\ &= F(b) - C - (F(a) - C) \\ &= F(b) - F(a) \end{aligned} $$

5. Thus what we observed in the introductory example holds in general.

6. *Fundamental Theorem of Calculus (Version 2):* If f is continuous on $[a, b]$ and F is an antiderivative of f, then

$$ \int_a^b f(t)\,dt = F(b) - F(a). $$

7. Note that, as we saw in the problem at the beginning of class, the Fundamental Theorem of Calculus is intuitively true when $f(t)$ is a rate of change. It is more surprising for a function $f(t)$ that NOT a rate of change.

5 References

- §23.3 in *Calculus: An Integrated Approach to Functions and Their Rates of Change.*