Math Xb Spring 2005
Average Value and Antiderivatives

April 18, 2005

1 Goals

- To be able to calculate the average value of a function

- To find simple antiderivatives and use these to calculate definite integrals

2 New Terms

- Average value of a function

- Antiderivative

- Indefinite Integral

3 Average Value

A dove flies at a velocity of $w(t) = 3t^2 + 4$ meters per second on the time interval $[1, 3]$. What is the average velocity of the dove during this time interval?

To find average velocity we first find the distance traveled during the time interval $[1, 3]$ and then divide by the time that elapsed, 2 seconds. To find the distance traveled, we must calculate the definite integral

$$\int_1^3 \left(3t^2 + 4\right) \, dt.$$

Note that the average velocity is not simply the average of the velocity at $t = 1$ and the velocity at $t = 3$.

This kind of calculation for average value of a function (in the example, a velocity function) works in general, no matter what kind of function we are given. If we want to find the average value of a function $f(x)$ on an interval $[a, b]$, we must find

$$\text{Average value of } f(x) = \frac{\text{Net change in } f(x)}{\text{Change in } x}$$
$$= \frac{\int_a^b f(x) \, dx}{b - a}$$

4 Indefinite Integrals and Antiderivatives

1. *Definition:* The symbol $\int f(x) \, dx$ stands for the entire family of functions that are antiderivatives of f. It is called the *indefinite integral.*

2. Recall: a definite integral is a limit of a sum but an antiderivative of f is a function whose derivative is f. It is the FTC that connects the antiderivative to the definite integral.

3. Note that if F is an antiderivative of f, then every antiderivative of f can be written as $F(x)+C$ for some constant C. Thus, $\int f(x) \, dx = F(x) + C$. The distinction between an antiderivative and an indefinite integral is that an indefininite integral is a whole family of functions, where as an antiderivative is just a single function. This is why the indefinite integral contains a "+C."

4. Since we know the derivatives of a lot of functions, we also know the antiderivatives to a lot of functions.

   (a) $\int k \, dx = kx + C$, where k is a constant

   (b) $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$

   (c) $\int e^x \, dx = e^x + C$

   (d) $\int b^x \, dx = \frac{b^x}{\ln b} + C$

   (e) $\int \sin x \, dx = -\cos x + C$

   (f) $\int \cos x \, dx = \sin x + C$

   (g) $\int \sec^2 x \, dx = \tan x + C$

   (h) $\int \csc^2 x \, dx = -\cot x + C$

   (i) $\int \sec x \tan x \, dx = \sec x + C$

   (j) $\int \csc x \cot x \, dx = -\csc x + C$

   (k) $\int \frac{1}{1+x^2} \, dx = \arctan x + C$

   (l) $\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$

   (m) $\int \frac{1}{x} \, dx = \ln |x| + C$

5. Note that the antiderivative of $\frac{1}{x}$ contains an absolute value. This is because ln is not defined for negative numbers. If $x > 0$, then the formula is just what we would expect. If $x < 0$, then $\ln x$ is undefined, but

$$\frac{d}{dx} \ln(-x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$$

Hence for $x < 0$, $\int \frac{1}{x} = \ln(-x)$. We can combine the result for $x < 0$ and $x > 0$ by using the absolute value.

6. Some of the differentiation formulas are easily translated into antidifferentiation formulas.

   (a) $\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$

   (b) $\int k f(x) \, dx = k \int f(x) \, dx$

7. Is it true that

$$\int f(x) g(x) \, dx = \left(\int f(x) \, dx \right) \left(\int g(x) \, dx \right)?$$

No!

5 References

- §24.2, 25.1 in *Calculus: An Integrated Approach to Functions and Their Rates of Change.*