Math Xb Spring 2005
Substitution I

April 20, 2005

1 Goals

- Be able to use "guess and check" to find antiderivatives of certain functions
- Understand the connection between the chain rule and the method of substitution
- Be able to use substitution to find certain antiderivatives and definite integrals

2 New Terms

- Substitution

3 Introduction: Guess-and-Check

Last time, we used what we know about derivative formulas to come up with a list of antiderivatives. So, for example, since we know that $\frac{d}{dx}\sin x = \cos x$, we get

$$\int \cos x = \sin x + C.$$

What can we do if we are asked to find an antiderivative and we don't know a corresponding derivative formula? One possibility is just to guess at the answer.

For example, find $\int \cos(6x)\,dx$. You might guess $\sin(6x)$. When you differentiate this, you get $6\cos(6x)$, but you were supposed to just get $\cos(6x)$. So, guess again – how about $\sin(6x)/6$? When you differentiate this, it works.

4 Discovering the Substitution Rule

1. The above problem is an example of the following pattern: if $\int f(x) = F(x) + C$, then $\int f(ax)\,dx = \frac{1}{a}F(ax) + C$.
2. Note how this comes from the corresponding differentiation rule
$$\frac{d}{dx}F(ax) = aF'(ax) = af(ax).$$
3. The above derivative is one example of using the is using the Chain Rule.
$$\frac{d}{dx}\left(F(u(x))\right) = F'(u(x)) \cdot u'(x).$$
4. What rule of antidifferentiation does this give us?
$$\int f(u(x)) \cdot u'(x)\,dx = F(u(x)).$$
This technique for antidifferentiating is known as *Substitution*.

5. For example, consider $\int 2x\sec(x^2)\,dx$. Let $f(x) = \sec x$ and $u(x) = x^2$. Then $F(x) = \tan x$ (the antiderivative of $\sec x$) and $u'(x) = 2x$. Thus

$$\int 2x\sec(x^2)\,dx = \int f(u(x)) \cdot u'(x)\,dx = F(u(x)) = \tan(x^2).$$

6. We usually make the substitutions by first identifying that there is some function "inside of" another function. We call this inside function u and write $u = u(x)$. Then we find $\frac{du}{dx} = u'(x)$ and look to see if our integrand has the form $f(u(x)) \cdot u'(x)\,dx$.

7. For example, consider $\int 4\cos 4x\,dx$. Let $u = 4x$. Then $\frac{du}{dx} = 4$ so $du = 4\,dx$. Thus,

$$\int 4\cos 4x\,dx = \int \cos u\,du = \sin u + C = \sin 4x + C.$$

8. We can also write the substitution rule as $u = u(x)$, $du = u'(x)dx$ and then this gives the formula

$$\int f(u(x)) \cdot u'(x)\,dx = \int f(u)\,du = F(u) + C.$$

   It may seem strange to go from $\frac{du}{dx} = 4$ to $du = 4\,dx$. We are treating $\frac{du}{dx}$ as if it were a fraction. It is not a fraction, but the notation we are using was chosen (in part) to that we can treat it as a fraction.

5 Substitution with Definite Integrals

1. We have two options when we are trying to use substitution to find definite integrals. We can
   (a) change the limits of integration from x-values to u-values during the substitution step, or
   (b) leave the limits of integration as x-values until the very end of the problem.

6 References

- §25.2 in *Calculus: An Integrated Approach to Functions and Their Rates of Change.*