Math Xb Spring 2005
Numeric Methods

April 25, 2005

1 Goals

- Be able to use Left-Hand, Right-Hand, Midpoint, or Trapezoidal Sums to approximate a definite integral
- Be able to find $|L_n - R_n|$ without calculating both L_n and R_n.
- For a function $f(x)$ which is monotonic on $[a, b]$, be able to figure out how many intervals you need to use for a R_n or L_n approximation to be within a given error

2 New Terms

- Monotonic
- Midpoint Sum
- Trapezoidal Sum

3 Motivating Problem

We've gone over one technique for finding antiderivatives of functions and there are others. Can we use our techniques find an antiderivative for every function?

If we give a fairly simple antiderivative like

$$\int e^{x^2}\,dx$$

to a powerful computing program like Mathematica, it gives the answer

$$\frac{1}{2}\sqrt{\pi}\mathrm{Erfi}(x)$$

and when you try to find out what Erfi is it tells you it's the function

$$\mathrm{Erfi}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

In other words, it doesn't give you an answer at all, it just tells you the answer is another integral.

There are many fairly innocent-looking functions whose antiderivatives cannot be written down as a combination of elementary functions. Given that this is true, what are we to do? We seem to have come up with a powerful tool that is no help to us at all for such functions.

Remember, however, that we already know that we can approximate definite integrals, even if we can't find antiderivatives and use the FTC.

4 Left and Right Hand Sums, with Error Estimates

We can estimate the integral

$$\int_0^3 e^{x^2}\,dx$$

using a right hand sum. But given that we have no way of finding the exact answer, how can we figure out how accurate our approximation is?

Let's restrict ourselves to functions which are *monotonic* on an interval $[a, b]$, that is, always decreasing or always increasing on the interval interval $[a, b]$. For such a function, the integral

$$\int_a^b f(x)\,dx$$

is trapped in between the Right-Hand Sum and the Left-Hand Sum.

This tells us that the *error* that comes from using the approximation R_n or L_n instead of the exact value of the integral must be less than or equal to the distance between R_n and L_n. This is

$$\begin{aligned}
|R_n - L_n| &= |(f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x) - \\
&\qquad (f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x))| \\
&= |f(x_n)\Delta x - f(x_0)\Delta x| \\
&= |f(x_n) - f(x_0)|\Delta x \\
&= |f(b) - f(a)|\Delta x
\end{aligned}$$

Remember that $\Delta x = (b - a)/n$.

5 Midpoint Sum and Trapezoidal Sum

Another method we have for approximating integrals, that is generally more accurate than Left- and Right-Hand Sums, is to use a Midpoint Sum. To use this method of approximation, we break up into n intervals like usual, but this time we don't take the left- or right-hand endpoint. Instead we take the midpoint of the interval.

The worksheet for this class steps through the Trapezoidal Sum.

6 References

- §26.1 in *Calculus: An Integrated Approach to Functions and Their Rates of Change.*