Math Xb Spring 2005
Review for Final Exam: Integration and Differential Equations

This is only a rough outline of the big ideas about integration and differential equations that you need to know for the final exam. For a complete list, see the review guide on the web site.
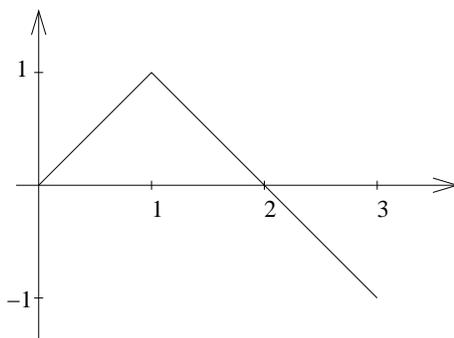
1 Integration

1.1 Definite Integral

1.1.1 Area = Net Change

If $v(t)$ is velocity (a rate of change) then $\int_a^b v(t)\,dt$ is distance travelled from $t = a$ to $t = b$ (that is, the *net change* in distance).

But $\int_a^b v(t)\,dt$ is also the signed area under the graph of $y = v(t)$.

Example: What is the distance travelled between 0 and 3 seconds, if the velocity, in meters per second, is graphed below?

1.1.2 Approximating integral

Remember that we defined the definite integral as a limit of a sum of rectangles. Thus a sum of rectangles is a good way to approximate an integral. You can use a right- or left-hand sum.
Example: Approximate the area under $y = \ln x$ from $x = 1$ to $x = 5$ using a right-hand sum. Is this answer and over- or under-approximation?

1.1.3 Properties: Magic with Integrals

The properties of integrals allow us to use incomplete information about an integral to find an integral or to set boundaries on how big or small an integral can be.
Example: If $\int_0^1 f(x)\,dx = 3$ and $\int_0^5 f(x)\,dx = -1$, what is $\int_1^5 f(x)\,dx$?

Example: If $0 \le g(x) \le 2$ for all x, find A and B such that

$$A \le \int_3^6 g(x)\,dx \le B.$$

1.2 Fundamental Theorem of Calculus (FTC)

1.2.1 Version 1: Derivative of the area function

The first version of the FTC says that

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

Example: If $y = f(t)$ is graphed below, where is $g(x) = \int_3^x f(t)\,dt$ increasing? Where is $g(x)$ concave down?

1.2.2   Version 2: Finding definite integrals

Given a velocity $v(t)$, if we want to find distance travelled from $t = 2$ to $t = 5$, we can first antidifferentiate $v(t)$ to get a possible distance function $d(t)$. Then distance travelled is $d(5) - d(2)$. This is the second version of the FTC. More formally, it says that if $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

1.3   Chain Rule

1.3.1   Pattern of inside function and it's derivative

The chain rule says that if $F(x)$ is an antiderivative of $f(x)$, then

$$\int f(g(x))g'(x)\,dx = F(g(x)).$$

Look for the pattern of an inside function and it's derivative when you are trying to apply the chain rule.
Example:

$$\int \cos(5x)5\,dx =$$

$$\int \cos(5x)\,dx =$$

$$\int \cos(x^2)2x\,dx =$$

$$\int \cos(x^2)x\,dx =$$

$$\int \cos(\sin x)\cos x\,dx =$$

$$\int \cos(\ln x)\tfrac{1}{x}\,dx =$$

$$\int \frac{\cos(\ln x)}{x}\,dx =$$

$$\int \frac{\cos(\sqrt{x})}{2\sqrt{x}}\,dx =$$

1.3.2   Don't be afraid to guess

Sometimes it is not obvious at all what should be the "inside" function, u. Don't be afraid to guess. Pick a u, find du, and see if it fits the problem. If not, pick another u.

Example: $\int \sec^2 x \tan^2 x\,dx$

1.3.3   Take care with substitution in definite integrals

Example: $\int_0^\pi \cos(5x)5\,dx.$

2   Differential Equations

2.1   Modeling with Diff Eqs

2.1.1   Proportional growth/decay

"The rate of growth of the population is proportional to the population."

"The rate of decay of a radioactive isotope is proportional to the amount of the isotope."

"The rate at which a drug is eliminated from the body is proportional to the amount of drug in the body."

All of these are examples of a rate of growth being proportional to a function. The differential equation for each of these examples will have the form $\frac{dA}{dt} = kA$, where $A(t)$ is the function we are trying to model. In each case the solution will be an exponential function $A(t) = A_0 e^{kt}$. This is one of the few kinds of differential equations that we know how to solve.

2.1.2 Rate in - rate out

Example: Suppose I am filling my above-ground pool, but it has spring a leak at the bottom. I'm filling the pool at a constant rate of 5 gallons per minute, but the rate at which the water is leaking out is proportional to the amount of water in the pool. Write a differential equation modeling this situation.

2.2 Solution or not?

To test whether a proposed solution to a differential equation really works, check that the proposed solution makes both sides of the equation equal.
Example: Which of the following is a solution to $\frac{dP}{dt} = kP - 60$?

- $P = 5e^{kt} - 60t$

- $P = 5e^{kt} + \frac{60}{k}$

2.3 Qualitative analysis

2.3.1 Do it when you can't solve

We can tell a lot about what happens to a solution of a differential equation over time, even if we can't actually find the solution. This is called qualitative analysis.

2.3.2 Equilibrium solutions

An equilibrium solution is a solution where nothing changes over time. This means that the solution is constant, or in other words that the derivative much be equal to zero. We can use the differential equation itself to find equilibrium solutions by setting the derivative equal to zero.
Example: What are the equilibrium solutions for the pool problem above $(dW/dt = 5 - kW)$ and what do they mean in terms of the pool?

2.3.3 Stable vs. Unstable

An equilibrium solution is stable if nearby solutions tend toward the equilibrium over time. An equilibrium solution is unstable if nearby solutions tend away from the equilibrium over time.
Example: Is the equilibrium for the pool problem stable or unstable?
Example: You have a bunny farm. The rate of growth of the population of bunnies is proportional to the population itself. The bunnies are being sold off at a continuous rate of 5 bunnies per week. A differential equation

$$\frac{dP}{dt} = kP - 5$$

models this situation. What is the equilibrium solution? Is it stable or unstable?
Example: Which slope field represents the bunnies and which represents the pool?