1 Implicit Differentiation

In logarithmic differentiation, we begin with an equation

$$y = f(x)$$

and then take the logarithm of both sides to get

$$\ln y = \ln f(x).$$

In this equation, y is not explicitly expressed as a function of x, but we are still able to differentiate both sides to find $\frac{dy}{dx}$.

Click or Clunk? (circle one)

If it clicks, what is "it"? If it clunks, what went wrong?

Logarithmic differentiation is a special case of **implicit differentiation**. The basic idea is that we can find $\frac{dy}{dx}$ even when we don't have an explicit equation $y = f(x)$. We do this by differentiating both sides of the an equation that relates x and y, applying the Chain Rule to differentiate terms involving y because y varies with x. We begin with an example.

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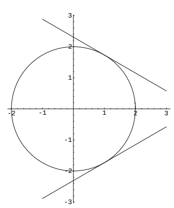
If it clicks, what is "it"? If it clunks, what went wrong?

1.1 Example:

Consider the circle of radius 2 centered at the origin.[a] It is given by the equation $x^2 + y^2 = 4$. Find the slope of the line tangent to the circle at the points $(1, \sqrt{3})$ and $(1, -\sqrt{3})$.



[a] This circle is the set of all points a distance 2 from the origin. If (x, y) is a point on this circle, then the distance formula tells us that $\sqrt{(x-0)^2 + (y-0)^2} = 2$. And conversely, if (x, y) satisfies the equation $\sqrt{x^2 + y^2} = 2$, then (x, y) is a point on the circle. Therefore, $x^2 + y^2 = 4$ is the equation of the circle.

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If it clicks, what is "it"? If it clunks, what went wrong?

Solution Although y is not a function of x, it can be expressed as two different functions of x.

$$y = \sqrt{4 - x^2} \text{ (the top semicircle), and} \quad (1)$$

$$y = -\sqrt{4 - x^2} \text{ (the bottom semicircle)} \quad (2)$$

One approach is to differentiate equation (1) to find the slope at $(1, \sqrt{3})$ and equation (2) to find the slope at $(1, -\sqrt{3})$. You can do this on your own and compare your answers with those below, but we're going to take a shortcut.

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If it clicks, what is "it"? If it clunks, what went wrong?

The equation $x^2+y^2=4$ *implicitly* gives a relationship between x and y. This means that the relationship between x and y is implied, instead of being directly expressed. Mathematically, instead of writing $y=f(x)$, we have an equation in which x and y can appear on either side of the equals sign (or even on both sides).

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If it clicks, what is "it"? If it clunks, what went wrong?

We are looking for $\frac{dy}{dx}$, the rate of change of y with respect to x. We differentiate both sides of the equation, treating y as if it were a function of x.

$$\frac{d}{dx}[x^2+y^2]=\frac{d}{dx}[4]$$

Just as we needed to use the Chain Rule to find $\frac{d}{dx}[\ln y]$ when we did logarithmic differentiation in the previous section, we need to use the Chain Rule here to evaluate $\frac{d}{dx}\left[y^2\right]$.

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If it clicks, what is "it"? If it clunks, what went wrong?

Because y depends on x, what we are really trying to find here is $\frac{d}{dx}(\text{mess})^2$. The Chain Rule tells us that the derivative of $(\text{mess})^2$ is $2(\text{mess})\cdot(\text{mess})'$. Thus when we differentiate y^2 we get $2y\frac{dy}{dx}$. Here is what we get when we differentiate both sides.

$$2x+2y\frac{dy}{dx}=0$$

Click or Clunk? (circle one)
If it clicks, what is "it"? If it clunks, what went wrong?

Now we solve for $\frac{dy}{dx}$.

$$\begin{aligned}2y\tfrac{dy}{dx} &= -2x\\ \tfrac{dy}{dx} &= \tfrac{-x}{y}.\end{aligned}$$

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If it clicks, what is "it"? If it clunks, what went wrong?

You may initially be startled that the formula for $\frac{dy}{dx}$ involves not only x, but y also. This happens because y is *not* a function of x. A given x-value may correspond to more than one y-value and therefore may have more than one slope associated with it. For example, $x=1$ at two points on the circle, $(1,\sqrt{3})$ and $(1,-\sqrt{3})$. We need to specify a value for y to know which point is meant. It thus makes sense that our formula for $\frac{dy}{dx}$ should involve both x and y.

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If it clicks, what is "it"? If it clunks, what went wrong?

Since we found $\frac{dy}{dx} = \frac{-x}{y}$, at $(1, \sqrt{3})$, the slope of the tangent line is $-1/\sqrt{3}$, while at $(1, -\sqrt{3})$ the slope of the tangent line is $1/\sqrt{3}$.
□

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If it clicks, what is "it"? If it clunks, what went wrong?

In the problem above, we found that $\frac{dy}{dx} = \frac{-x}{y}$. We might like to express this answer entirely in terms of x. We can do this. If we are looking at a point on the top semicircle where $y = \sqrt{4 - x^2}$, we can write $\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}$; if we were looking at a point on the bottom semicircle where $y = -\sqrt{4 - x^2}$, we can write $\frac{dy}{dx} = \frac{-x}{-\sqrt{4-x^2}}$.

Click or Clunk? (circle one)
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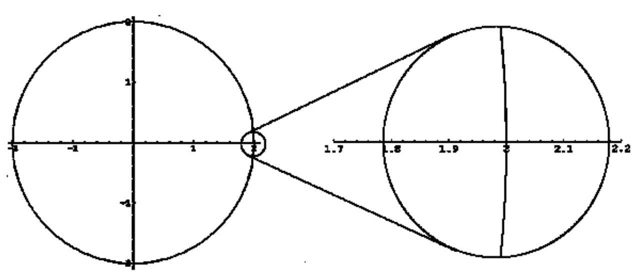
Another thing to notice about our solution is that $\frac{dy}{dx}$ is undefined when $y = 0$. This corresponds to the points $(2, 0)$ and $(-2, 0)$ on the circle. Get out your magnifying glass and take a good look at the curve in the immediate vicinity of each of these points. No matter how much the curve is magnified, it *never* looks like the graph of a function.



Click or Clunk? (circle one)
If it clicks, what is "it"? If it clunks, what went wrong?

Intuitively speaking, $\frac{dy}{dx}$ is defined at a point P if, under magnification, the curve around P looks like the graph of a differentiable function.

Click or Clunk? (circle one)
If it clicks, what is "it"? If it clunks, what went wrong?

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2 The Process of Implicit Differentiation

1. Decide which variable you want to differentiate with respect to (x if you want $\frac{dy}{dx}$, t if you want $\frac{dy}{dt}$, etc.).
2. Differentiate both sides of the equation with respect to that variable.

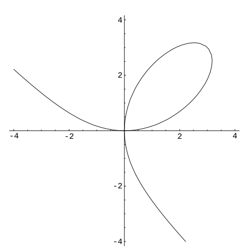
   Remember the Chain Rule! Suppose you are differentiating with respect to t. Distinguish between quantities that vary with t (treating them as functions of t) and those that are independent of t (treating them as constants). In particular, when looking for $\frac{dy}{dt}$, think of y as a function of t.
3. Solve to find a formula for the desired derivative. If you only want to know the derivative at a specific point, you can substitute in the coordinates of that point before solving for the derivative you're trying to find.

Click or Clunk? (circle one)

If it clicks, what is "it"? If it clunks, what went wrong?

2.1 Example:

The curve shown below is called the *folium of Descartes*. It is the set of all points satisfying the equation $x^3 + y^3 = 6xy$. The point $(3, 3)$ lies on this curve; when we substitute $x = 3$ and $y = 3$ into the equation, both sides are equal to 54. Find the slope of a tangent line to $x^3 + y^3 = 6xy$, first in general, and then at the point $(3, 3)$.



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If it clicks, what is "it"? If it clunks, what went wrong?

<u>Practice</u>: Attempt this problem in the space below before you move on to the next page. If you get stuck, first check that you are following the outlined procedure. If you are still stuck, look at the solution on the next page for a hint.

Solution We need to determine $\frac{dy}{dx}$. Notice that we do not have an explicit formula for y in terms of x. In fact, a glance at the graph shows that y is not a function of x. However, if we magnify the curve right around the point $(3,3)$, it does look like the graph of a function. Thus we can use implicit differentiation, treating y as if it were a function of x and applying the Chain Rule where necessary.

Click or Clunk? (circle one)

If it clicks, what is "it"? If it clunks, what went wrong?

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(6xy)$$ Differentiate each side with respect to x.

$$3x^2+3y^2\frac{dy}{dx} = 6y+6x\frac{dy}{dx}$$

Note that we have to use both the product rule and the chain rule to differentiate the left hand side. This is because $6xy$ is a product of two functions of x: $6x$ and y.

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If it clicks, what is "it"? If it clunks, what went wrong?

Now we need to solve for $\frac{dy}{dx}$. The equation we have is linear in $\frac{dy}{dx}$, so we use the standard strategy for solving linear equations.

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If it clicks, what is "it"? If it clunks, what went wrong?

$$3x^2+3y^2\frac{dy}{dx} = 6y+6x\frac{dy}{dx}$$ Original equation.

$$3y^2\frac{dy}{dx}-6x\frac{dy}{dx} = 6y-3x^2$$ Bring $\frac{dy}{dx}$ terms to one side.

$$\frac{dy}{dx}(3y^2-6x) = 6y-3x^2$$ Factor out $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x}$$ Divide through to isolate $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{2y-x^2}{y^2-2x}$$

Click or Clunk? (circle one)

If it clicks, what is "it"? If it clunks, what went wrong?

We now have a formula for $\frac{dy}{dx}$; this expression makes sense, provided $y^2 \neq 2x$.[a]

[a] At the points on the folium of Descartes where $y^2 = 2x$ the derivative is undefined. y does not look like a differentiable function of x, no matter how much you magnify the area around the point in question. For example, look at the point $(0, 0)$.

Click or Clunk? (circle one)

If it clicks, what is "it"? If it clunks, what went wrong?

To find the slope of the line tangent to the folium of Descartes at the point $(3, 3)$, substitute $x = 3$ and $y = 3$ into the expression for $\frac{dy}{dx}$, obtaining $\frac{dy}{dx} = -1$. Looking at the shape of the graph at the point $(3, 3)$, this seems like a reasonable answer.

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If it clicks, what is "it"? If it clunks, what went wrong?

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3 Differentiating "with respect to" a Particular Variable

When using implicit differentiation, we have to know what variables are dependent on other variables. We will differentiate with respect to the independent variable, and use the chain rule on the dependent variables. Suppose, for instance, that we will differentiate both sides of an equation with respect time t (the independent variable). We must establish which quantities vary with respect to time and which do not. We'll clarify this by means of an example.

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If it clicks, what is "it"? If it clunks, what went wrong?

3.1 Example:

Let P be the pressure under which a gas is kept, V be the volume of the gas, and T be temperature measured on the absolute (or Kelvin) scale. The combined gas law tells us that

$$\frac{PV}{T} = K,$$

where K is a constant.

Click or Clunk? (circle one)

If it clicks, what is "it"? If it clunks, what went wrong?

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1. Suppose temperature is kept constant. Express the rate of change of volume with respect to pressure.

2. Suppose volume is kept constant. Express the rate of change of temperature with respect to pressure.

Click or Clunk? (circle one)

If it clicks, what is "it"? If it clunks, what went wrong?

3. Suppose volume is kept constant but pressure and temperature change with time. What is the relationship between the change in pressure with respect to time and the change in temperature with respect to time?

4. Suppose pressure, temperature, and volume all change with time. How are these rates of change related?

Click or Clunk? (circle one)
If it clicks, what is "it"? If it clunks, what went wrong?

Solution

1. In this part of the problem P is the independent variable and V is the dependent variable. In other words, we think of V as a function of P. T is treated as a constant in this part of the problem. K is also a constant. We can emphasize this by writing

$$\frac{P}{T}V(P) = K. \tag{3}$$

Click or Clunk? (circle one)
If it clicks, what is "it"? If it clunks, what went wrong?

We want to find $\frac{dV}{dP}$. One option is to solve for V in terms of P to get

$$V(P) = \frac{KT}{P} = KTP^{-1}.$$

To find out how V varies with P we differentiate with respect to P (remember that K and T are both constants), obtaining

$$\frac{dV}{dP} = KT(-1)P^{-2} = \frac{-KT}{P^2}.$$

Click or Clunk? (circle one)
If it clicks, what is "it"? If it clunks, what went wrong?

Another option is to differentiate the original equation (3) implicitly. Note that we'll need to use the product rule and the chain rule on the left hand side.

$$\begin{aligned}\tfrac{d}{dP}\left[\tfrac{P}{T}V(P)\right] &= \tfrac{d}{dP}K, \text{ so}\\ \tfrac{1}{T}\left(P\tfrac{dV}{dP} + V(P)\right) &= 0\\ \tfrac{P}{T}\tfrac{dV}{dP} &= \tfrac{-V(P)}{T}.\end{aligned}$$

Click or Clunk? (circle one)
If it clicks, what is "it"? If it clunks, what went wrong?

Solving for $\frac{dV}{dP}$ gives

$$\frac{dV}{dP} = \frac{-V(P)}{P}.$$

This is equivalent to the previous answer, because $V(P) = \frac{KT}{P}$.

Click or Clunk? (circle one)
If it clicks, what is "it"? If it clunks, what went wrong?

2. Here we are thinking of P as the independent variable and looking for $\frac{dT}{dP}$. Now V is treated as a constant and T as a function of P. We can emphasize this by writing $\frac{PV}{T(P)} = K$.

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If it clicks, what is "it"? If it clunks, what went wrong?

The simplest strategy here is to solve for T explicitly and then differentiate with respect to P.

$$T(P) = \frac{PV}{K} = \frac{V}{K}P$$

where V and K are both constant. Therefore $\frac{dT}{dP} = \frac{V}{K}$.

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If it clicks, what is "it"? If it clunks, what went wrong?

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3. For this part the independent variable is the time, t. T and P are both dependent variables. Both T and P vary with time but V and K are constants. It may be helpful to write

$$\frac{P(t) \cdot V}{T(t)} = K$$

to emphasize that P and T vary with time.

Click or Clunk? (circle one)
If it clicks, what is "it"? If it clunks, what went wrong?

We will differentiate the equation with respect to t. Our job will be easier if we rewrite the equation as

$$V \cdot P(t) = K \cdot T(t)$$

because then we won't need the quotient rule.

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If it clicks, what is "it"? If it clunks, what went wrong?

Differentiating with respect to t gives

$$V\frac{dP}{dt} = K\frac{dT}{dt}$$

or

$$\frac{dP}{dt} = \frac{K}{V}\frac{dT}{dt}.$$

Note that we we can eliminate both V and K from this last equation, because $V = K\frac{T}{P}$.

$$\frac{dP}{dt} = K \cdot \frac{P}{KT} \cdot \frac{dT}{dt}$$

or

$$\frac{dP}{dt} = \frac{P}{T}\frac{dT}{dt}.$$

Click or Clunk? (circle one)
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4. Again we think of time t as the independent variable, but this time P, V, and T are all dependent variables. Only K is constant. It may be helpful to write

$$V(t) \cdot P(t) = K \cdot T(t).$$

We differentiate with respect to t, and we need to use the Product Rule on the left-hand side.

$$V(t) \cdot \frac{dP}{dt} + \frac{dV}{dt} \cdot P(t) = K \cdot \frac{dT}{dt}$$

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Note that we can eliminate the constant K completely, replacing it by $\frac{PV}{T}$.

$$V(t) \cdot \frac{dP}{dt} + \frac{dV}{dt} \cdot P(t) = \frac{PV}{T} \cdot \frac{dT}{dt}.$$

Observe that if we know P,V, and T at a certain instant, then knowing two of the rates of change at that moment allows us to determine the third.

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Click or Clunk? (circle one)
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Remark Notice that the notation V' can be ambiguous. Is it $\frac{dV}{dP}$ or $\frac{dV}{dt}$? We use it only when it is clear from context what is meant. The notation V'' can also be ambiguous. The second derivative of V with respect to P, $\frac{d}{dP}\left(\frac{dV}{dP}\right)$ can be written as $\frac{d^2V}{dP^2}$. The second derivative of V with respect to t, $\frac{d}{dt}\left(\frac{dV}{dt}\right)$ can be written as $\frac{d^2V}{dt^2}$.

Click or Clunk? (circle one)
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