Name: Solutions

Math Xb Final Exam Part One
Friday, May 21, 2004

Please circle your section:

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Will Adams (CA)	Jennie Schiffman (CA)	Ashley Hinerman-Mulroy (CA)
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Part One Scores

Problem Number	Possible Points	Score
1	16	
2	16	
Total	32	

Final Exam Scores

	Possible Points	Score
Part One	32	
Part Two	68	
Total	100	

Directions—Please read carefully! You *are not allowed* to use a calculator or any other aids on this part of the exam. Be sure to write neatly—illegible answers will receive little or no credit. When you are finished with this part of the exam, you may turn it in to the proctor and receive the second part of the exam. Once you have turned in this first part of the exam, you may not look at it again, so be sure you have finished it completely before turning it in. The second part of the exam consists of eight problems, and you may use a calculator on the second part of the exam. **Good luck!**

1. Find the derivative of each of the following functions. You need not simplify your answers.

(a) $f(x) = 2x^2 - x^{3/2} + x - \pi + \dfrac{1}{x}$

$$f'(x) = 4x - \frac{3}{2}x^{1/2} + 1 - x^{-2}$$

(b) $f(x) = 2x\sqrt{x}\tan x$ $= 2x^{3/2}\tan x$

$$f'(x) = 2 \cdot \frac{3}{2}x^{1/2}\tan x + 2x^{3/2}\sec^2 x$$

(c) $f(x) = \dfrac{\sin x + \cos x}{\ln x}$

$$f'(x) = \frac{(\cos x - \sin x)\ln x - (\sin x + \cos x)\cdot \frac{1}{x}}{(\ln x)^2}$$

(d) $f(x) = e^{\sec^2 x}$

$$f'(x) = e^{\sec^2 x} \cdot 2\sec x \cdot \tan x \sec x$$

2. Evaluate each of the following integrals.

(a) $\int \frac{1}{1+x^2}\,dx$

$= \arctan x + C$

(b) $\int \tan x\,dx$

$= \int \frac{\sin x}{\cos x}\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$= -\int \frac{du}{u}$

$= -\ln|u| + C \quad = \boxed{-\ln|\cos x| + C}$

(c) $\int_1^2 xe^{x^2}, dx$

Method 1

$$\int_1^2 xe^{x^2}dx = \int_{u=1}^{u=4} \frac{1}{2}e^u du$$

$$u = x^2$$
$$du = 2x\,dx$$

$$= \frac{1}{2}e^u \Big|_{u=1}^{u=4}$$

$$\boxed{= \frac{1}{2}(e^4 - e^1)}$$

Method 2

$$\int_1^2 xe^{x^2}dx = \int_{x=1}^{x=2} \frac{1}{2}e^u du = \frac{1}{2}e^u\Big|_{x=1}^{x=2} = \frac{1}{2}e^{x^2}\Big|_{x=1}^{x=2} = \frac{1}{2}(e$$

(d) $\int x\sqrt{2x-1}\,dx$

$$u = 2x-1 \longrightarrow x = \frac{u+1}{2}$$
$$du = 2\,dx$$

$$\int x\sqrt{2x-1}\,dx = \int \frac{u+1}{2}\cdot\sqrt{u}\cdot\frac{1}{2}du$$

$$= \frac{1}{4}\int\left(u^{3/2}+u^{1/2}\right)du$$

$$= \frac{1}{4}\left[\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right]+C$$

$$= \boxed{\frac{1}{10}u^{5/2}+\frac{1}{6}u^{3/2}+C}$$

Name:______________________________

Math Xb Final Exam Part Two
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Part Two Scores

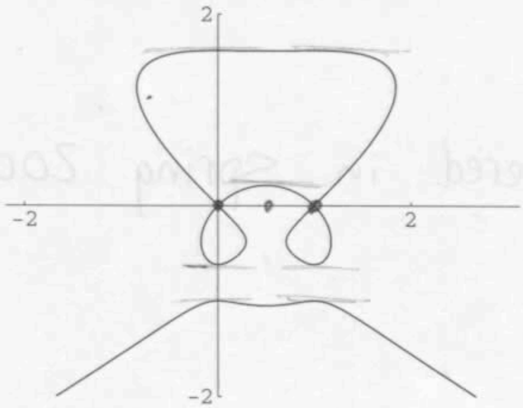
Problem Number	Possible Points	Score
1	8	
2	8	
3	8	
4	8	
5	8	
6	12	
7	8	
8	8	
Total	68	

Directions—Please read carefully! You *are allowed* to use a calculator on this part of the exam, but no other aids are allowed. Read each problem carefully and make sure to answer the specific questions asked. Some questions ask you to justify or explain your answers. You must do so on to receive full credit on these questions. Be sure to write neatly—illegible answers will receive little or no credit. You may not return to the first part of the exam. **Good luck!**

1. The curve with equation

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$$

has been likened to a bouncing wagon, as you can see in the graph below. Find the exact x-coordinates of the points at which the curve has horizontal tangent lines. Justify your answer using calculus. (Estimating the x-coordinates from the graph is not sufficient.)



Horizontal tangents happen when $y' = 0$

Implicitly diff both sides of equation:

$$6y^2y' + 2yy' - 5y^4y' = 4x^3 - 6x^2 + 2x$$

$$y'(6y^2 + 2y - 5y^4) = 4x^3 - 6x^2 + 2x$$

$$y' = \frac{4x^3 - 6x^2 + 2x}{6y^2 + 2y - 5y^4}$$

$y' = 0$ where $4x^3 - 6x^2 + 2x = 0$

$$2x(2x^2 - 3x + 1) = 0$$

$$2x(2x - 1)(x - 1) = 0$$

$$\boxed{x = 0, \ x = \tfrac{1}{2}, \ x = 1}$$

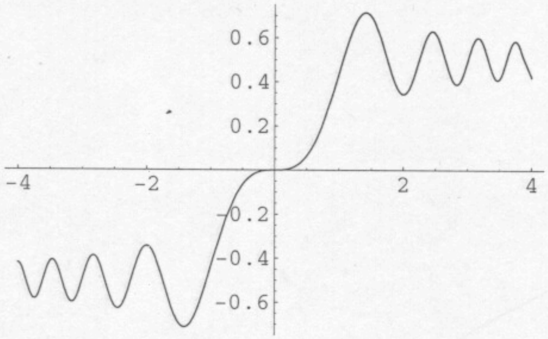
2. A student plans to obtain her Ph.D. and join the faculty at Harvard ten years from now. She would like to save enough money to buy an apartment in Cambridge at that time. She plans to put a fixed amount of money in her savings account *at the beginning of each month* for ten years. Assuming an annual interest rate of 6 percent *compounded monthly*, how much should she deposit each month in order to have 1 million dollars available for her apartment immediately *after her last deposit*?

(topic not covered in Spring 2005)

4. The Fresnel function

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$$

is named after the French physicist Augustin Fresnel (1788–1827), who is famous for his work in optics. This function first appeared in Fresnel's theory of the diffraction of light.



(a) Find $S'(x)$. Justify your answer.

By the fundamental theorem of calculus, version 1,

$$S'(x) = \sin\left(\frac{\pi x^2}{2}\right)$$

(b) Use L'Hôpital's Rule to evaluate $\lim_{x\to 0} \frac{S(x)}{x^3}$.

$\lim_{x\to 0} S(x) = 0$ & $\lim_{x\to 0} x^3 = 0$

So we can use L'Hospital's Rule:

$$\lim_{x\to 0} \frac{S(x)}{x^3} = \lim_{x\to 0} \frac{S'(x)}{3x^2} = \lim_{x\to 0} \frac{\cos\left(\frac{\pi x^2}{2}\right)\cdot \frac{2\pi x}{2}}{6x}$$

$\sin(0) = 0$ so still have form $\frac{0}{0}$, use L'Hospital's rule again

can apply L'Hospital's Rule again since form is $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{-\sin\left(\frac{\pi x^2}{2}\right)\pi^2 x^2 + \cos\left(\frac{\pi x^2}{2}\right)\pi}{6}$$

$$= \boxed{\frac{\pi}{6}}$$

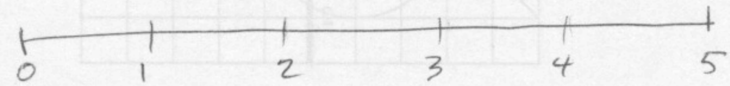
5. The rate at which the world's oil is being consumed is increasing. Suppose that the rate (measured in billions of barrels per year) is given by the function $r(t)$, where t is measured in years and $t = 0$ represents January 1, 1990.

(a) Write a definite integral that represents the total quantity of oil used between *the start of* 1990 and *the start of* 1995.

$$\int_0^5 r(t)\,dt$$

(b) Suppose that $r(t) = 32e^{0.05t}$. Find the approximate value for the definite integral from part (a) using a right-hand sum with $n = 5$ subintervals.

Interval: $[0, 5]$ with $n = 5$ so $\Delta x = \frac{5-0}{5} = 1$



$$R_5 = r(1)\Delta x + r(2)\Delta x + r(3)\Delta x + r(4)\Delta x + r(5)\Delta x$$
$$= \left(32e^{.05} + 32e^{.1} + 32e^{.15} + 32e^{.2} + 32e^{.25}\right)\cdot 1$$
$$\approx 186.359$$

(c) Interpret each of the five terms in the sum from part (b) in terms of oil consumption.

$32e^{.05}$ is oil used in 1990 (approximate)

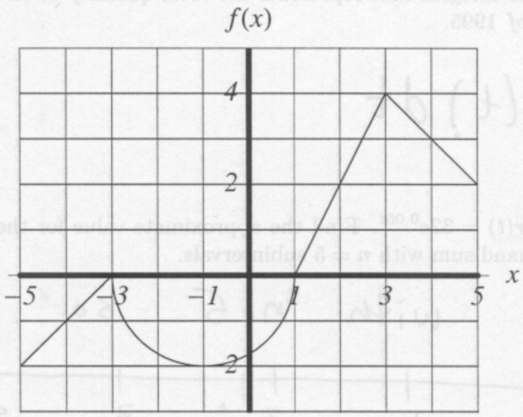
$32e^{.1}$ is oil used in 1991 (approximately)

$32e^{.15}$ is approximate oil used in 1992

$32e^{.2}$ " " 1993

$32e^{.25}$ " " 1994

6. Let $F(x) = \int_{-3}^{x} f(t)\,dt$, where f is the function whose graph is given below. Note that the graph of f is made up of straight lines and a semicircle. Also note that -3 is the lower limit of integration in the definition of F.



(a) Identify the x-values of all critical points of F in the interval $(-5, 5)$.

Critical points are where $F'(x) = 0$ and $F'(x) = f(x)$ by the FTC so critical points are where $x = -3$ and $x = 1$

(b) On what interval(s) in $(-5, 5)$ is F decreasing? Justify your answer.

$F'(x) = f(x)$ by the FTC so

$F(x)$ decreasing where $F'(x) = f(x) < 0$, so on intervals $(-5, 3)$ and $(3, 1)$

(c) At what x-values in the interval $(-5, 5)$, if any, does F have a local maximum? Justify your answer.

$F'(x) = f(x)$ never goes from being positive to negative, which is what we need for a local max, so there are no local maxima for $F(x)$

(d) At what x-values in the interval $(-5, 5)$, if any, does F have a local minimum? Justify your answer.

$F'(x) = f(x)$ goes from being negative to being positive at $x = 1$, thus $F(x)$ has local minimum at $x = 1$.

(e) Find each of the following values. If a value is not defined, explain why not.

i. $F(1)$

$$= \int_{-3}^{1} f(t)\,dt = -\begin{pmatrix}\text{area of semicircle} \\ \text{of radius 2}\end{pmatrix} = -4\pi$$

ii. $F'(1)$

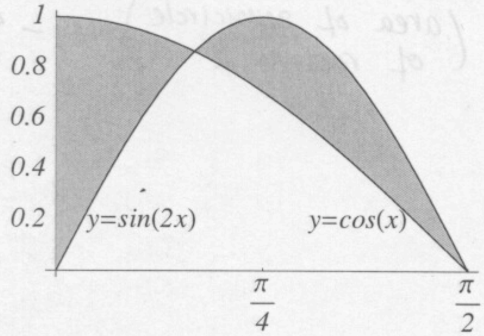
$$= f(1) = 0$$

iii. $F''(1)$

$= f'(1)$, which is undefined

($f(t)$ is not smooth at $t=1$)

7. Find the area of the region enclosed by the curves $y = \cos x$ and $y = \sin 2x$ between $x = 0$ and $x = \frac{\pi}{2}$. That is, find the area of the shaded region graphed below.



Not covered in Spring 2005

8. Psychologists interested in learning theory study learning curves. A learning curve is the graph of a function $P(t)$, the performance of someone learning a skill as a function of training time t. The derivative dP/dt represents the rate at which performance improves.

If M is the maximum level of performance of which the learner is capable, then

$$\frac{dP}{dt} = k(M - P)$$

is one reasonable model for learning.

(a) Interpret the differential equation given above for dP/dt in the context of learning curves. That is, explain in words what the differential equation tells us about the rate at which performance improves.

The performance of a person is proportional to the diffuence between their maximum level of performance & their current performance.

(b) Explain why the differential equation given above might be seen as a reasonable model of learning.

As a person gets closer to his or her maximum capacity, the learning is getting increasingly difficult, thus the rate of learning is slowing down.