Name: Solutions.

Math Xb Midterm Examination I
Tuesday, March 15, 2005

Please circle your section:

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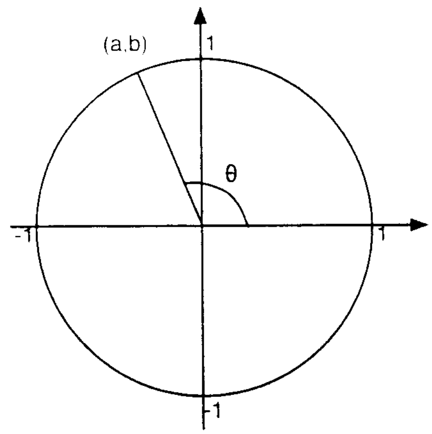
Problem Number	Possible Points	Score
1	8	
2	10	
3	9	
4	10	
5	12	
6	10	
7	12	
8	10	
9	10	
10	9	

Directions—Please Read Carefully! You have two hours to take this midterm. To receive full credit on a problem, you will need to justify your answers carefully—**unsubstantiated answers will receive little or no credit** (except if the directions for that question specifically say no justification is necessary). Please be sure to write neatly—**illegible answers will receive little or no credit**. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers.

Make sure to use correct mathematical notation. Any answer in decimal form must be accurate to two decimal places, unless otherwise specified. You may use a calculator on this exam, but no other aids are allowed. **Good Luck!**

1. (8 points) Consider the following diagram of the unit circle with an angle θ measured in radians. For each of the following, answer true or false (no explanation required.)



(a) $\cos\theta$ is positive. F

(b) $\tan\theta$ is negative. T

(c) $\sin(\theta/2)$ is negative. F

(d) $\tan(2\theta)$ is positive (assume the picture is drawn to scale). T

(e) $a^2 + b^2 = 1$. T

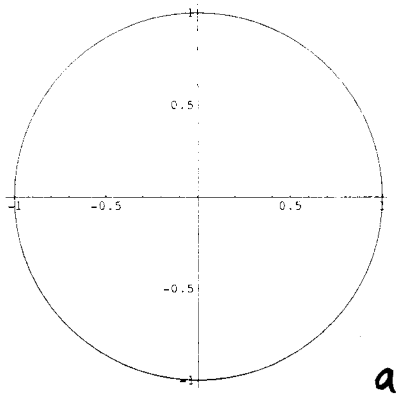
(f) The length of the arc on the circle from $(-a, -b)$ to (a, b) is π. T

(g) $\sec(\theta + 9\pi) = 1/a$. F

(h) $\cot(\theta + 9\pi) = a/b$. T

2. (10 points) For each of the following graphs, find the equation of the tangent line at the given point.

(a) $x^2 + y^2 = 1$, at the point $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$.



Diff both sides wrt x

$$2x + 2y\frac{dy}{dx} = 0$$

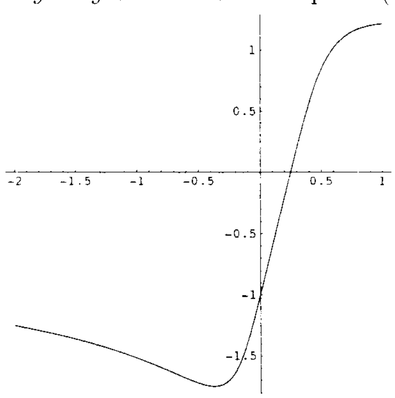
$$\frac{dy}{dx} = \frac{-2x}{2y}$$

at $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$,

$$\frac{dy}{dx} = \frac{-2(-\frac{1}{2})}{2(-\frac{\sqrt{3}}{2})} = -\frac{1}{\sqrt{3}}$$

so $\boxed{y + \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}\left(x + \frac{1}{2}\right)}$

(b) $x^2y^3 + y + 1 = 4x$, at the point $(0, -1)$.



Diff both sides wrt x

$$2xy^3 + 3x^2y^2\frac{dy}{dx} + \frac{dy}{dx} = 4$$

$$(3x^2y^2 + 1)\frac{dy}{dx} = 4 - 2xy^3$$

so $\frac{dy}{dx} = \frac{4-2xy^3}{3x^2y^2+1}$

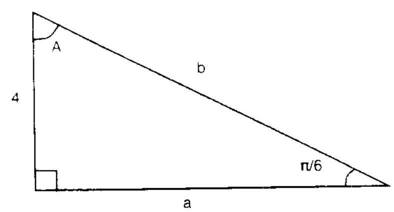
at $(0,-1)$, $\frac{dy}{dx} = \frac{4-0}{0+1} = 4$

so $y + 1 = 4(x - 0)$

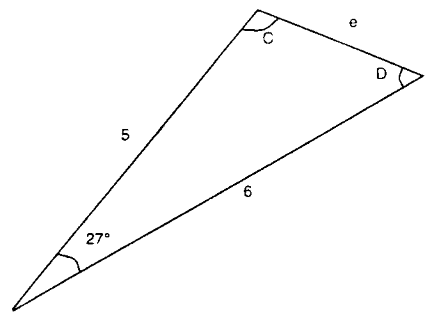
$\boxed{y = 4x - 1}$

3. (9 points) Consider the following triangles.

1



2.



(a) Find the angle A in radians and the lengths of the sides a and b.

$\boxed{A = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}}$

$\sin\frac{\pi}{6} = \frac{4}{b}$

$\frac{1}{2} = \frac{4}{b}$ so $\boxed{b = 8}$

$\cos\frac{\pi}{6} = \frac{a}{b} = \frac{a}{8} \Rightarrow \frac{\sqrt{3}}{2} = \frac{a}{8} \Rightarrow \boxed{a = 4\sqrt{3}}$

(b) Find the angles C and D in degrees, and the length of the side e.

$e^2 = 5^2 + 6^2 - 2(5)(6)\cos 27^\circ$

$\boxed{e \approx 2.7458}$

$\frac{\sin D}{5} = \frac{\sin 27^\circ}{2.7458}$

$\sin D = .8267$

$\boxed{D \approx 55.76^\circ}$ or $.97$ rad

$27 + D + C = 180$

so $\boxed{C = 97.24^\circ}$

(c) Which triangle has the largest area? You must justify your answer to receive credit. (Note that the triangles may not be drawn to scale)

$\Delta 1$: $A = \frac{1}{2}(4)(a) = \frac{1}{2}(4)(4\sqrt{3})$

≈ 13.86

$\Delta 2$: $A = \frac{1}{2}(5)(6)\sin 27^\circ \approx 6.81$

so $\Delta 1$ has largest area

4. (10 points) A tanker has spilled oil of the coast of Boston. Oil eating bacteria are gobbling 5 ft^3 of oil per hour (note that oil is no longer spilling out of the tanker). The oil slick can be modeled as a very flat right circular cylinder, where the height of the cylinder is the thickness of the slick. When the radius of the cylinder is 500 ft, the height of the cylinder is 0.01 feet and decreasing at the rate of 0.001 ft per hour .

(a) What is the rate of change of the radius of the cylinder at this time? You should note whether the radius is increasing or decreasing.

(b) What is the rate of change of the area of the top of the oil slick at this time? You should note whether the area is increasing or decreasing.

You might find the following formulas useful: The volume of a cylinder is $V = \pi r^2 h$. The area of a circle is $A = \pi r^2$.

(a) $V = \pi r^2 h$

$$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$$

$$-5 = 2\pi(500)\frac{dr}{dt}(.01) + \pi(500)^2(-.001)$$

$$-5 = 10\pi \frac{dr}{dt} - 250\pi$$

$$\text{so } \frac{dr}{dt} = \frac{250\pi + 5}{10\pi} \approx 25.16 \text{ ft/hr}$$

(b) $A = \pi r^2$

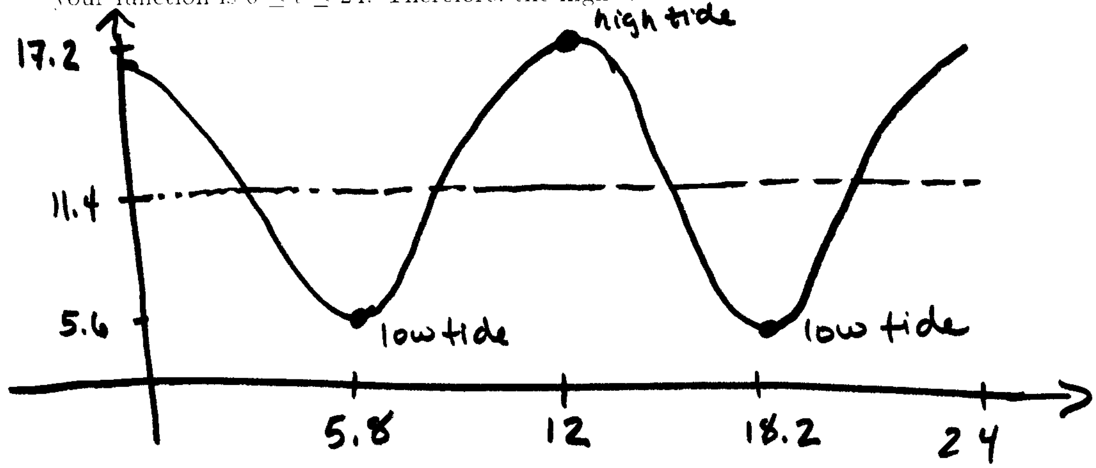
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dA}{dt} \approx 2\pi(500)(25.16)$$

$$\approx 78040 \text{ ft/hr}$$

5. (12 points) In a tidal river, the time between high tide and low tide is 6.2 hours. At high tide the depth of the water is 17.2 feet, while at low tide, the depth is 5.6 feet. Assume that the water depth is a trigonometric function of time.

(a) Sketch the graph of the depth of the water over time if there is a high tide at 12:00 noon. Label your graph indicating the high and low tide. Assume that $t = 0$ is midnight and the domain of your function is $0 \le t \le 24$. Therefore, the high tide occurs at $t = 12$.



(b) Write a function for the graph that you sketched in part (a).

$$f(t) = 11.4 + 5.8\cos\left(\frac{2\pi}{12.4}(t-12)\right)$$

(c) A boat requires a depth of 8 feet to sail, and is docked at 12:00 noon. What is the latest time in the afternoon that the boat can sail? Your answer should be accurate to the nearest minute.

$$11.4 + 5.8\cos\left(\frac{2\pi}{12.4}(t-12)\right) = 8$$

$$\cos\left(\frac{2\pi}{12.4}(t-12)\right) = \frac{-3.4}{5.8}$$

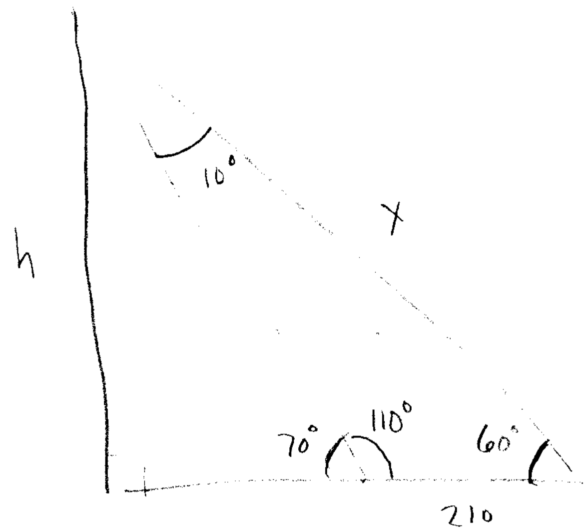
using inverse cosine

$$\frac{2\pi}{12.4}(t-12) = 2.197$$

$$t \approx 16.34$$

so $\boxed{4:20\text{ pm}}$

6. (10 points) A tall monument cannot be measured by simply taking a long measuring tape and dropping it down from the top of the monument to the ground. Instead, we can use trigonometry. To measure the Eiffel Tower in Paris, a person stands away from the base and measures an angle of elevation to the top of the tower to be 60°. Moving 210 feet closer, the angle of elevation to the top is 70°. How tall is the Eiffel Tower?



$$\frac{\sin 10^\circ}{210} = \frac{\sin 110^\circ}{x}$$

$$\Rightarrow \quad x = \frac{210 \cdot \sin 110^\circ}{\sin 10^\circ} \approx 1136 \text{ ft}$$

(17.678 in Radians)

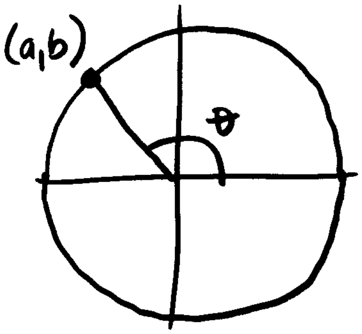
$$\sin 60^\circ = \frac{h}{x}$$

$$\Rightarrow h = x \sin 60^\circ$$

$$= \frac{210 \cdot \sin 110^\circ \cdot \sin 60^\circ}{\sin 10^\circ} \approx 984 \text{ ft}$$

7. (12 points) Suppose that the point (a, b) lies on the unit circle. That is, $a^2 + b^2 = 1$.

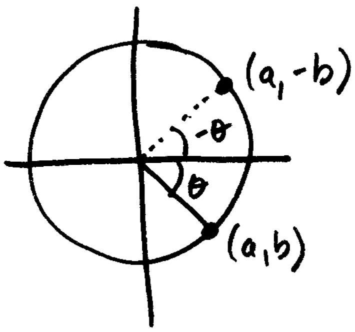
(a) If $a < 0$ and $b > 0$, express $\sin(\cos^{-1} a)$ in terms of a and b.



$\cos^{-1} a = \theta$

$\sin(\cos^{-1} a) = \sin\theta = b$

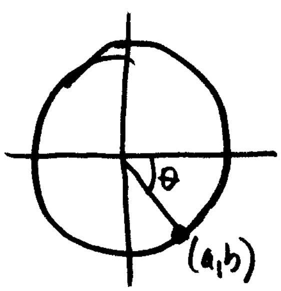
(b) If $a > 0$ and $b < 0$, express $\sin(\cos^{-1} a)$ in terms of a and b.



$\cos^{-1} a = -\theta$

$\sin(-\theta) = -b$

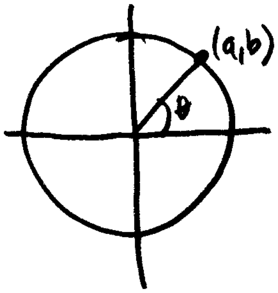
(c) If $a > 0$ and $b < 0$, express $\tan(\sin^{-1} b)$ in terms of a and b.



$\sin^{-1}(b) = \theta$

$\tan\theta = \dfrac{b}{a}$

(d) If $a > 0$ and $b > 0$, express $\sin(2\cos^{-1} a)$ in terms of a and b.



$\cos^{-1} a = \theta$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2ab$$

8. (10 points) Let $g(x) = f(x)^x$. Find $g'(3)$, given that you know $f(3) = e^5$ and $f'(3) = e^5 + 1$. Give an exact value – do not give a decimal approximation.

$$g(x) = f(x)^x$$

$$\ln g(x) = \ln f(x)^x$$

$$\ln g(x) = x \ln f(x)$$

$$\frac{g'(x)}{g(x)} = \ln f(x) + x \frac{f'(x)}{f(x)}$$

$$g'(x) = g(x)\left[\ln f(x) + x \frac{f'(x)}{f(x)}\right]$$

$$g'(3) = g(3)\left[\ln f(3) + 3 \frac{f'(3)}{f(3)}\right]$$

$$= f(3)^3\left[\ln e^5 + 3 \frac{e^5+1}{e^5}\right]$$

$$\boxed{g'(3) = e^{15}\left[5 + \frac{3(e^5+1)}{e^5}\right]}$$

$$(8e^5 + 3)e^{10}$$

$$8e^{15} + 3e^{10}$$

9. (10 points)

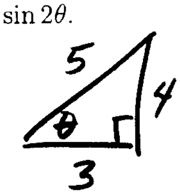
(a) Express the following in terms of $\cos x$ and $\cos y$. Be sure to show all the steps of your work.

$$\cos(x+y)+\cos(x-y)$$

$\cos x \cos y - \sin x \sin y + \cos x \cos y + \sin x \sin y$

$= 2 \cos x \cos y$

(b) If $\sin\theta = \frac{4}{5}$ and θ is an acute angle (less than $90°$), find an exact value for $\sin 2\theta$.



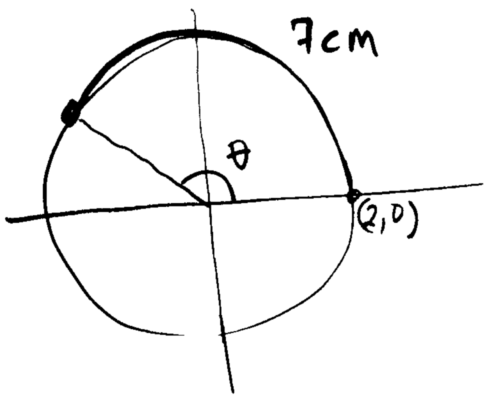
$\sin 2\theta = 2 \sin\theta \cos\theta$

$= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$

$= \frac{24}{25}$

[illegible] ([illegible] points) A bug is crawling around a circle with equation $x^2 + y^2 = 4$. It starts at the point $(2, 0)$ and proceeds counter-clockwise. It carefully measures the distance it has travelled and stops after crawling 7 cm.

(a) Through what angle *in degrees* has the bug travelled?



arclength $= 2\theta$

$7 = 2\theta$

$7/2 = \theta$ in radians

$\frac{7}{2} \cdot \frac{180 \text{ deg}}{\pi \text{ rad}} = \frac{630}{\pi}$ degrees

$\approx 200.54^\circ$

(b) What are the approximate x and y coordinates of the bug?

$x = 2\cos\theta$

$= 2\cos\frac{7}{2}$

$\approx -.70$

$y = 2\sin\theta$

$= 2\sin\frac{7}{2}$

≈ -1.87

(c) Imagine drawing a line from the current position of the bug to the origin. What is the slope of that line?

Slope $= \tan\theta = \tan\frac{7}{2} \approx .37$