Name: SOLUTIONS

Math Xb Midterm Examination I
Tuesday, March 16, 2004

Please circle your section:

Derek Bruff	Ken Chung	Thomas Judson
Will Adams (CA)	Jennie Schiffman (CA)	Ashley Hinerman-Mulroy (CA)
MWF 10–11	MWF 10–11	MWF 11–12

Kazem Mahdavi	Derek Bruff
Maria van Wagenberg (CA)	Connie Zong (CA)
MWF 11–12	MWF 12–1

Problem Number	Possible Points	Score
1	8	
2	8	
3	8	
4	8	
5	8	
6	6	
7	10	
8	8	
9	8	
10	8	
11	6	
12	6	
13	8	
Total	100	

Directions—Please Read Carefully! You have two hours to take this midterm. Make sure to use correct mathematical notation. Any answer in decimal form must be accurate to two decimal places, unless otherwise specified. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You do not have to answer the problems in any particular order, so move to another problem if you find you are stuck or spending too much time on a single problem. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit (except if the directions for that question specifically say no justification is necessary). Please be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. You may use a calculator on this exam, but no other aids are allowed. **Good Luck!**

1. Let $g(x) = x^{\sqrt{x}} f(x)$. Using logarithmic differentiation, find $g'(4)$ assuming that f is a differentiable function, $f(4) = 1$, and $f'(4) = \frac{1}{2}$.

$$g(x) = x^{\sqrt{x}} f(x) \longrightarrow g(4) = 4^{\sqrt{4}} \cdot f(4) = 4^2 \cdot 1 = 16$$

$$\ln g(x) = \ln\left(x^{\sqrt{x}} f(x)\right) = \sqrt{x} \ln x + \ln f(x)$$

$$\frac{g'(x)}{g(x)} = \frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}} + \frac{f'(x)}{f(x)}$$

$$g'(x) = g(x)\left(\frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}} + \frac{f'(x)}{f(x)}\right)$$

$$g'(4) = g(4)\left(\frac{\sqrt{4}}{4} + \frac{\ln 4}{2\sqrt{4}} + \frac{f'(4)}{f(4)}\right)$$

$$= 16\left(\frac{1}{2} + \frac{\ln 4}{4} + \frac{1/2}{1}\right)$$

$$= 16\left(1 + \frac{\ln 4}{4}\right) = \boxed{16 + 4\ln 4}$$

2. Find an equation of the tangent line to the curve

$$x^3y^4 - 5 = x^3 - x^2 + y$$

at the point $(2, -1)$.

$$x^3 \cdot 4y^3 y' + 3x^2 \cdot y^4 = 3x^2 - 2x + y'$$

$$4x^3y^3y' - y' = 3x^2 - 2x - 3x^2y^4$$

$$y'(4x^3y^3 - 1) = 3x^2 - 2x - 3x^2y^4$$

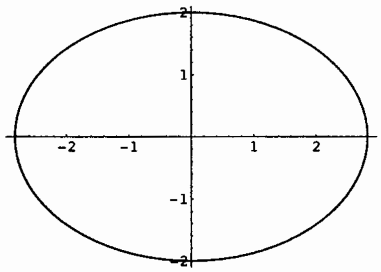
$$y' = \frac{3x^2 - 2x - 3x^2y^4}{4x^3y^3 - 1}$$

$$y'\Big|_{(2,-1)} = \frac{3 \cdot 2^2 - 2 \cdot 2 - 3 \cdot 2^2(-1)^4}{4 \cdot 2^3(-1)^3 - 1}$$

$$= \frac{12 - 4 - 12}{-32 - 1} = \frac{4}{33}$$

$$\boxed{y + 1 = \frac{4}{33}(x - 2)}$$

3. At what points on the ellipse $x^2 + 2y^2 = 8$ is the slope of the tangent line equal to -1?



$$2x + 4yy' = 0$$

$$4yy' = -2x$$

$$y' = \frac{-2x}{4y} = \frac{-x}{2y}$$

$$\frac{-x}{2y} = -1$$

$$x = 2y \quad \text{and} \quad x^2 + 2y^2 = 8$$

$$(2y)^2 + 2y^2 = 8$$

$$4y^2 + 2y^2 = 8$$

$$6y^2 = 8$$

$$y^2 = 4/3$$

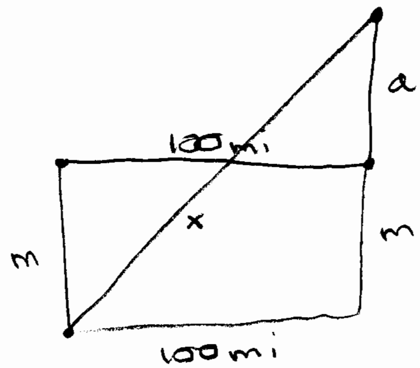
$$y = \pm 2/\sqrt{3}$$

$$y = 2/\sqrt{3} \Rightarrow x = 2y = 4/\sqrt{3}$$

$$y = -2/\sqrt{3} \Rightarrow x = 2y = -4/\sqrt{3}$$

$$\boxed{(4/\sqrt{3},\ 2/\sqrt{3}) \ \& \ (-4/\sqrt{3},\ -2/\sqrt{3})}$$

4. At noon, Ashley leaves Boston, heading north in her car at a constant rate of 60 miles per hour. At the same time, Mary-Kate is in her car, 100 miles west of Boston, heading south at a constant rate of 50 miles per hour. At what rate is the distance between them changing at 4 p.m.?



$\frac{da}{dt} = 60$ mph

$\frac{dm}{dt} = 50$ mph

$\frac{dx}{dt} = ?$ when $t = 4$

when $t=4$

$a = 240$

$m = 200$

$x = \sqrt{100^2 + (200+240)^2}$

≈ 451

$$100^2 + (a+m)^2 = x^2$$

$$\cancel{2}(a+m)\left(\frac{da}{dt} + \frac{dm}{dt}\right) = \cancel{2}x\frac{dx}{dt}$$

$$(240+200)(60+50) = 451 \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{440 \cdot 110}{451} \approx \boxed{107.3 \text{ mph}}$$

5. Consider the following geometric sum.

$$3 + 2 + \frac{4}{3} + \frac{8}{9} + \cdots + \frac{256}{2187}$$

(a) Explain why the sum given above is geometric.

Each term is 2/3 of the previous term.

common ratio $r = 2/3$

(b) How many terms does the sum have?

$256 = 2^8$ so $2 + \frac{4}{3} + \frac{8}{9} + \ldots + \frac{256}{2187}$

$= 2^1 + \frac{2^2}{3} + \frac{2^3}{9} + \ldots + \frac{2^8}{2187}$

gives 8 terms

$\Rightarrow$ $\boxed{\text{9 terms total}}$

(c) Express the sum using summation notation.

$$\sum_{n=0}^{8} 3 \cdot \left(\frac{2}{3}\right)^n$$

(d) Find the numeric value of the sum.

$$\frac{3 - \frac{256 \cdot 2}{2187 \cdot 3}}{1 - \frac{2}{3}} = \boxed{\frac{19171}{2187}} \approx 8.77$$

6. Find a rational number (a number of the form $\frac{a}{b}$, where a and b are integers) that corresponds to the repeating decimal $0.0\overline{13} = 0.0131313\cdots$.

$$0.0\overline{13} = 0.0131313\cdots$$

$$= 0.013 + 0.00013 + 0.0000013 + \cdots$$

$$= \frac{13}{1000} + \frac{13}{100000} + \frac{13}{10000000} + \cdots$$

$$= \frac{13}{1000} + \frac{13}{1000}\cdot\frac{1}{100} + \frac{13}{1000}\cdot\left(\frac{1}{100}\right)^2 + \cdots$$

$$= \frac{\frac{13}{1000}}{1-\frac{1}{100}} = \boxed{\frac{13}{990}}$$

$\ulcorner = 2\cdot 3 + 2\cdot 3^2 + \dots + 2\cdot 3^{12}$

7. Which of the following are equal to $\sum_{n=1}^{12} 2\cdot 3^n$?

(a) $\sum_{n=0}^{11} 2\cdot 3^n$ $= 2\cdot 1 + 2\cdot 3 + \dots 2\cdot 3^{11}$ NO

(b) $\sum_{n=0}^{11} 2\cdot 3^{n+1}$ $= 2\cdot 3 + 2\cdot 3^2 + \dots + 2\cdot 3^{12}$ YES

(c) $\sum_{n=3}^{3^{12}} 2n$ $= 2\cdot 3 + 2\cdot 4 + 2\cdot 5 + \dots + 2\cdot 3^{12}$ NO

(d) $\sum_{k=2}^{12} 2\cdot 3^{k-1}$ $= 2\cdot 3 + 2\cdot 3^2 + \dots + 2\cdot 3^{11}$ NO

(e) $3^{13} - 3$

$$\sum_{n=1}^{12} 2\cdot 3^n = \frac{2\cdot 3 - 2\cdot 3^{13}}{1-3} = \frac{2\cdot 3 - 2\cdot 3^{13}}{-2}$$

$$= 3^{13} - 3$$ YES

8. Three years ago, the Texas Rangers signed baseball player Alex Rodriguez to a record ten-year, \$252 million contract. Rodriguez was recently traded to the New York Yankees with about \$175 million on the remaining seven years of his contract.

Suppose the Yankees were to pay Rodriguez \$175 million in equal yearly payments over the next seven years starting one year from today. Also suppose that the Yankees are going to pay him out of a bank account bearing 5% interest compounded annually. How much would the Yankees have to put in the account today in order to pay Rodriguez?

$$x_t(1.05)^t = \frac{175}{7} = 25 \text{ million}$$

$$x_t = \frac{25}{(1.05)^t}$$

$$\sum_{t=1}^{7} x_t = \sum_{t=1}^{7} \frac{25}{(1.05)^t} = \frac{\frac{25}{1.05} - \frac{25}{1.05^8}}{1 - \frac{1}{1.05}}$$

$$\approx \boxed{\$144.66 \text{ million}}$$

9. The Bay of Fundy in Canada has the highest tides in the world. The tidal range (that is, the difference between between low and high tides) is 16 meters. There are two high tides every 24.8 hours. We can use this information to model the height h of the water (in meters above sea level) as a function of time t (in hours since midnight on a particular day) by the function

$$h(t) = D + A\cos(B(t - C)).$$

(a) What should the value of A be?

$16 = \text{amplitude} \cdot 2 = 2A$

$\Rightarrow A = 8$

(b) What should the value of B be?

$\frac{2\pi}{B} = \text{period} = 12.4 \text{ hrs}$

$\Rightarrow B = \frac{2\pi}{12.4}$

(c) What is the physical meaning of C?

$h(C) = D + A\cos(B \cdot 0) = D + A$

$\Rightarrow$ high tide occurs C hours after midnight

(d) What is the physical meaning of D?

D is the average height of the tides, halfway between low tide and high tide

10. Find all solutions to the following equation.

$$3 = 2\cos^2 x + 3\sin x$$

$$3 = 2(1-\sin^2 x) + 3\sin x$$

$$3 = 2 - 2\sin^2 x + 3\sin x$$

$$0 = -2\sin^2 x + 3\sin x - 1$$

$$0 = 2\sin^2 x - 3\sin x + 1$$

$$0 = (2\sin x - 1)(\sin x - 1)$$

$\sin x = 1/2$ or $\sin x = 1$

$x = \frac{\pi}{6} + 2\pi n$

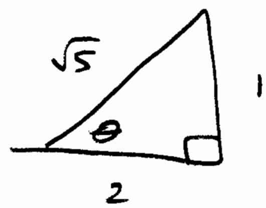
$\frac{5\pi}{6} + 2\pi n$ where n is any integer

$\pi/2 + 2\pi n$

11. Find the exact value of each of the following.

(a) $\sin\left(\underbrace{\tan^{-1}\left(\frac{1}{2}\right)}_{\theta}\right)$

$\tan\theta = 1/2$



$\Rightarrow \sin\theta = 1/\sqrt{5}$

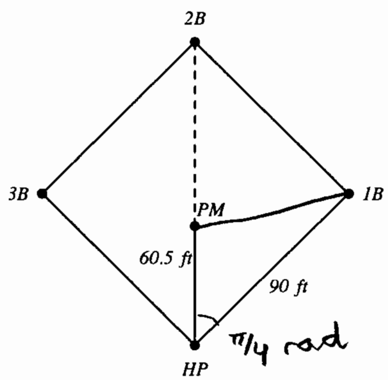
(b) $\cos\left(\tan^{-1}\left(\frac{1}{2}\right)\right)$

$\cos\theta = 2/\sqrt{5}$

(c) $\sin\left(2\tan^{-1}\left(\frac{1}{2}\right)\right) = \sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(1/\sqrt{5}\right)\left(2/\sqrt{5}\right) = 4/5$

12. A baseball diamond is a square with sides of length 90 feet, as shown below. The pitcher's mound is located 60.5 feet from home plate. Approximate the distance from the pitcher's mount to first base. (Note that the pitcher's mound is *not* halfway between home plate and second base. It is a little closer to home plate.)



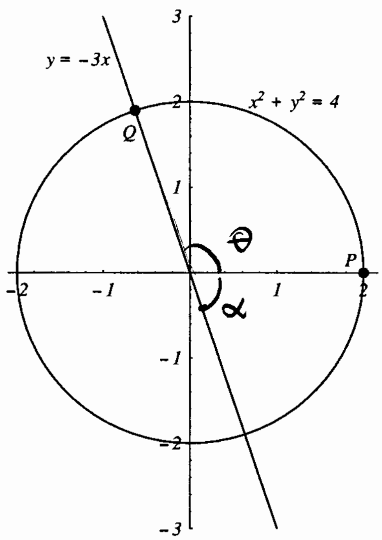
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$= (60.5)^2 + 90^2 - 2(60.5)(90)\cos \pi/4$$

$$\approx 4059.86$$

$$\Rightarrow \quad c \approx \boxed{63.72 \text{ ft}}$$

13. Let P be the point $(2, 0)$. Let Q be the point in the second quadrant where the line $y = -3x$ and the circle $x^2 + y^2 = 4$ intersect. If you started at point P and walked counter-clockwise around the circle to the point Q, how far would you travel?



$\tan\theta$ = slope of line $y = -3x$

$\Rightarrow \tan\theta = -3$

$\alpha = \tan^{-1}(-3) \approx -1.25$ rad

$\theta = \alpha + \pi$

$\approx -1.25 + \pi \approx 1.89$

arc length $= \theta \cdot$ radius

$\approx 1.89(2) \approx \boxed{3.79 \text{ units}}$