17.1 Introductory Example

4. $y = x^{x^2} = e^{x^2 \ln x} \Rightarrow y' = e^{x^2 \ln x}[2x \ln x + x^2 \frac{1}{x}] = x^{x^2}[2x \ln x + x]$

17.2 Logarithmic Differentiation

1. (a) $f(x) = 2x^x \Rightarrow \ln(f(x)) = \ln(2x^x) = \ln 2 + x \ln x$

$\Rightarrow \frac{1}{f(x)} f'(x) = (1) \ln x + x \frac{1}{x} = \ln x + 1 \Rightarrow f'(x) = [\ln x + 1] f(x) = [\ln x + 1] 2x^x$

(b) $\ln(f(x)) = \ln(5(x^2+1)^x) = \ln 5 + x \ln(x^2+1) \Rightarrow \frac{f'(x)}{f(x)} = 0 + (1)\ln(x^2+1) + x \frac{1}{(x^2+1)}(2x)$

$= \ln(x^2+1) + \frac{2x^2}{x^2+1} \Rightarrow f'(x) = [\ln(x^2+1) + \frac{2x^2}{x^2+1}] f(x) = [\ln(x^2+1) + \frac{2x^2}{x^2+1}] 5(x^2+1)^x$

(c) $\ln(f(x)) = \ln((2x^4+5)^{3x+1}) = (3x+1)\ln(2x^4+5) \Rightarrow \frac{f'(x)}{f(x)} = 3\ln(2x^4+5) + (3x+1)\frac{8x^3}{2x^4+5}$

$\Rightarrow f'(x) = (2x^4+5)^{3x+1}[3\ln(2x^4+5) + (3x+1)\frac{8x^3}{2x^4+5}]$

5. $y = f(x)g(x) \Rightarrow \ln y = \ln(f(x)g(x)) = \ln(f(x)) + \ln(g(x))$

$\Rightarrow \frac{y'}{y} = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \Rightarrow y' = \left(\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\right) f(x)g(x) \Rightarrow y' = f'(x)g(x) + g'(x)f(x)$

17.3 Implicit Differentiation

2. $x^2 + y^2 = 25 \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y} \Rightarrow \frac{dy}{dx}|_{(4,-3)} = \frac{-4}{-3} = \frac{4}{3}$

$x^2 + 4y^2 = 25 \Rightarrow 2x + 8y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{4y} \Rightarrow \frac{dy}{dx}|_{(4,-\frac{3}{2})} = \frac{-4}{4(-\frac{3}{2})} = \frac{2}{3}$

The slope is larger at (4, -3). You can see the same result if you carefully sketch the graphs.

4. $(x-2)^2 + (y-3)^2 = 25$. When $x = 6$, $16 + (y-3)^2 = 25 \Rightarrow (y-3)^2 = 9$

$\Rightarrow y - 3 = \pm 3 \Rightarrow y = 0, 6$. From implicit differentiation $2(x-2) + 2(y-3)\frac{dy}{dx} = 0$

At (6, 0), $2(4) + 2(-3)\frac{dy}{dx}|_{(6,0)} = 0 \Rightarrow \frac{dy}{dx}|_{(6,0)} = \frac{4}{3}$

At (6, 6), $2(4) + 2(3)\frac{dy}{dx}|_{(6,6)} = 0 \Rightarrow \frac{dy}{dx}|_{(6,6)} = \frac{-4}{3}$

5. $x^3 + y^3 - 3x^2y^2 + 1 = 0 \Rightarrow 3x^2 + 3y^2\frac{dy}{dx} - 3(2xy^2 + x^2 2y\frac{dy}{dx}) + 0 = 0$; so at (1,1)

$3 + 3\frac{dy}{dx} - 3(2 + 2\frac{dy}{dx}) = 0 \Rightarrow 3 + 3\frac{dy}{dx} - 6 - 6\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -1$

Hence the tangent line to the curve at (1,1) is $y - 1 = -1(x-1) \Rightarrow y = -x + 2$

7. $3x^2 + 6xy + 8y^2 = 8 \Rightarrow 6x + 6y + 6x\frac{dy}{dx} + 16y\frac{dy}{dx} = 0 \Rightarrow (6x+16y)\frac{dy}{dx} = -6x - 6y$. We cannot divide by $6x + 16y$ when it is equal zero, hence vertical tangent lines occur when $6x + 16y = 0$ or $y = \frac{-6x}{16} = \frac{-3}{8}x$. We need points to satisfy this as well as be on the original curve, hence we substitute for y, $3x^2 + 6x(-\frac{3}{8}x) + 8(-\frac{3}{8}x)^2 = 8 \Rightarrow 3x^2 - \frac{9}{4}x^2 + \frac{9}{8}x^2 = 8 \Rightarrow \frac{15}{8}x^2 = 8 \Rightarrow x = \pm\frac{8}{\sqrt{15}}$

Now for $x = \frac{8}{\sqrt{15}}$, $y = -\frac{3}{8}(\frac{8}{\sqrt{15}}) = -\frac{3}{\sqrt{15}}$. Similarly for $x = -\frac{8}{\sqrt{15}}$, $y = \frac{3}{\sqrt{15}}$.

Hence the tangents lines are vertical at $\left(\frac{8}{\sqrt{15}}, -\frac{3}{\sqrt{15}}\right)$ and $\left(-\frac{8}{\sqrt{15}}, \frac{3}{\sqrt{15}}\right)$.

10. $2(x^2+y^2)^2 = 25(x^2-y^2) \Rightarrow 4(x^2+y^2)(2x + 2y\frac{dy}{dx}) = 25(2x - 2y\frac{dy}{dx})$. At point (-3, 1)

$4(9+1)(-6 + 2\frac{dy}{dx}) = 25(-6 - 2\frac{dy}{dx}) \Rightarrow -240 + 80\frac{dy}{dx} = -150 - 50\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{9}{13}$.