Math Xb Gateway Exam: Algebra and Derivatives - Practice Problems

- *You may not use a calculator on this exam.*
- *Show all of your work. You will be graded on the entire process, not just the final answer.*
- *Unless stated otherwise, simplify and factor all of your answers as much as possible.*

The first ten problems are solved by using the Chain Rule.

1. Find the derivative of $f(x) = (x^2 + x + 3)^3$.

   SOLUTION: $3(x^2 + x + 3)^2(2x + 1)$

2. Find the derivative of $f(x) = \sqrt{e^{x^2+x}}$.

   SOLUTION: $\frac{1}{2}(e^{x^2+x})^{-1/2}e^{x^2+x}(2x + 1)$

3. Find the derivative of $f(x) = (x^{11} + e^x + e^{x^2})^{99}$.

   SOLUTION: $99(x^{11} + e^x + e^{x^2})^{98}(11x^{10} + e^x + 2xe^{x^2})$

4. Find the derivative of $f(x) = e^{\sqrt{x^7+x+x^{-1}}}$.

   SOLUTION: $e^{\sqrt{x^7+x+x^{-1}}}(\frac{1}{2})(x^7 + x + x^{-1})^{-1/2}(7x^6 + 1 - x^{-2})$

5. Find the derivative of $f(x) = (5x^5 + 3x^2 + 1)^{-5}$.

   SOLUTION: $-5(5x^5 + 3x^2 + 1)^{-6}(25x^4 + 6x)$

6. Find the derivative of $f(x) = \sqrt{e^{\sqrt{e^x}}}$.

   SOLUTION: $\frac{1}{2}(e^{\sqrt{e^x}})^{-1/2}e^{\sqrt{e^x}}(\frac{1}{2})\sqrt{e^x}$

7. Find the derivative of $f(x) = (7x^{99} + 98x)^{1000001}$.

   SOLUTION: $1000001(7x^{99} + 98x)^{1000000}(693x^{98} + 98)$

8. Find the derivative of $f(x) = e^{e^{e^x}}$.

   SOLUTION: $e^{e^{e^x}}e^{e^x}e^x$

9. Find the derivative of $f(x) = (\frac{1}{x} + \frac{1}{x^2})^{-3}$.

   SOLUTION: $-3(\frac{1}{x} + \frac{1}{x^2})^{-4}(-\frac{1}{x^2} - \frac{2}{x^3})$

10. Find the derivative of $f(x) = \frac{1}{3}(x^4 + x^3 + x^2 + x + 1)^3 + \frac{1}{2}(x^4 + x^3 + x^2 + x + 1)^2 + (x^4 + x^3 + x^2 + x + 1) + e^{x^4+x^3+x^2+x+1} + 1$.

SOLUTION: $\left((x^4+x^3+x^2+x+1)^2+(x^4+x^3+x^2+x+1)+1+e^{x^4+x^3+x^2+x+1}\right)(4x^3+3x^2+2x+1)$

11. Find all solutions to: $\frac{x}{2x+9} < \frac{1}{x+3}$.

SOLUTION: The key to these inequality problems is that you usually cannot multiply both sides by anything involving a variable. The reason for this is because multiplying by a negative reverses the inequality sign (e.g. $2 < 5$, but $-2 > -5$), and the variable may or may not be a negative number.

In particular, we might be tempted to multiply both sides of the inequality by $2x+9$. However, we cannot do this, because we do not know if $2x+9$ is negative. For instance, if $x = 1$, $2x+9$ is positive, but if $x = -10$, $2x+9$ is negative.

Since we do not know the sign (positive or negative) of $2x+9$ and $x+3$, we cannot use the strategy of multiplying to get rid of denominators. Instead, we will subtract to bring everything to one side of the inequality:

$\frac{x}{2x+9} - \frac{1}{x+3} < 0$

We can further combine the fractions to get:

$\frac{x^2+3x-(2x+9)}{(2x+9)(x+3)} < 0$
$\frac{x^2+x-9}{(2x+9)(x+3)} < 0$

This has the huge advantage of having a 0 on one side of the inequality. This is nice because 0 is the exact cut-off between the positive numbers and the negative numbers. So instead of thinking numerically, we can now think about this problem as "which values of x make the fraction negative."

To figure out when the fraction is negative, we simply need to determine when each of factors are negative. Once we have that, we will use the fact that "a negative times a negative is a positive." Hence, when we have an odd number of factors of the fraction being negative, the entire fraction will be negative. When there are an even number of factors being negative, then the fraction is positive.

So let's look at the factors. $2x+9$ is positive if $x > \frac{9}{2}$, and negative if $x < \frac{9}{2}$. So $\frac{9}{2}$ is the cut-off point.

The factor $x+3$ is positive if $x > -3$ and negative if $x < -3$. So 3 is the cut-off point for $x+3$.

Finally, we will consider x^2+x-9. It would be great if we could factor this into two linear factors, because then we could readily see where the factors are positive and negative. However, this is not easy. Instead, we will use our knowledge of parabolas.

The graph of x^2+x-9 is a parabola that opens upward, since the coefficient on the quadratic term is positive. Therefore, it is going to be positive most of the time. We can determine the roots of the parabola to be $x = \frac{-1\pm\sqrt{37}}{2}$ by using the quadratic formula. The only way to graph a parabola that opens up with these roots is to have the graph be negative if $\frac{-1-\sqrt{37}}{2} < x < \frac{-1+\sqrt{37}}{2}$, and positive elsewhere. So $x^2 + x - 9$ goes from positive to negative at $\frac{-1-\sqrt{37}}{2}$, and from negative to positive at $\frac{-1+\sqrt{37}}{2}$.

We know that if $x < -\frac{9}{2}$, then $2 + 9$ and $x + 3$ will be negative, and $x^2 + x - 9$ will be positive. Then we have a positive number in the numerator of $\frac{x^2+x-9}{(2x+9)(x+3)}$, and two negative numbers being multiplied together in the denominator. The two negatives cancel to give a positive number, so we know that $\frac{x^2+x-9}{(2x+9)(x+3)}$ is positive if $x < -\frac{9}{2}$.

Between $-\frac{9}{2}$ and $\frac{-1-\sqrt{37}}{2}$, the only thing that changes is that $2x + 9$ becomes positive. Then we have two positives and a negative, which means that $\frac{x^2+x-9}{(2x+9)(x+3)}$ is negative on this interval.

Between $\frac{-1-\sqrt{37}}{2}$ and -3, the only thing that changes is that $x^2 + x - 9$ becomes negative. Then we have one positive and two negatives, which means that $\frac{x^2+x-9}{(2x+9)(x+3)}$ is positive on this interval.

Between -3 and $\frac{-1+\sqrt{37}}{2}$, the only thing that changes is that $x+3$ becomes positive. Then we have two positives and one negative, which means that $\frac{x^2+x-9}{(2x+9)(x+3)}$ is negative on this interval.

If $x > \frac{-1+\sqrt{37}}{2}$, then everything is positive, so the entire fraction is positive.

We can now answer the question: "When is $\frac{x^2+x-9}{(2x+9)(x+3)}$ negative?" In other words, "when is $\frac{x^2+x-9}{(2x+9)(x+3)} < 0$?"

The answer is:

$(-\frac{9}{2}, \frac{-1-\sqrt{37}}{2}) \cup (-3, \frac{-1+\sqrt{37}}{2})$

12. Find all solutions to: $\frac{x}{x^2+4} < \frac{1}{x-1}$.

    SOLUTION: $(-\infty, -4) \cup (1, \infty)$

13. Find all solutions to: $\frac{6x}{x+2} > -\frac{x+1}{x-1}$.

    SOLUTION: $(\infty, -2) \cup (1, \infty)$

14. Find all solutions to: $\frac{3x^2+1}{x-1} > 2x$.

    SOLUTION: $(1, \infty)$

15. Find all solutions to: $\frac{x}{x+3} > -\frac{x+6}{2x+1}$.

    SOLUTION: $(-\infty, -3) \cup (-\frac{1}{2}, \infty)$

16. Find all solutions to: $\frac{x^2+1}{x-6} \leq \frac{x}{6}$.

SOLUTION: $(-\infty, 6)$

17. Find all solutions to: $\frac{5x}{x^2+x+1} \leq -\frac{1}{3x+5}$.

    SOLUTION: $(-\infty, -\frac{5}{3}) \cup [\frac{-13-\sqrt{153}}{16}, \frac{-13+\sqrt{153}}{16}]$

18. Find all solutions to: $\frac{2x+1}{x-1} \leq \frac{x+1}{2x-1}$.

    SOLUTION: $(\frac{1}{2}, 1)$

19. Find all solutions to: $\frac{3x+2}{x+2} \geq \frac{6x-1}{x-6}$.

    SOLUTION: $[\frac{-37-\sqrt{1149}}{6}, -2) \cup [-37 + \sqrt{11496}, 6)$

20. Find all solutions to: $\frac{x+1}{x} \geq -\frac{x}{x+1}$.

    SOLUTION: $(-\infty, -1) \cup (0, \infty)$

21. Find the derivative of $g(x) = (100x^2 + 9)^{-9}(5x^4 - 32x + 3)^5$.

    SOLUTION: To find the derivative, you simply use the product rule:

    $$(-9)(100x^2 + 9)^{-10}(200x)(5x^4 - 32x + 3)^5 + 5(5x^4 - 32x + 3)^4(20x^3 - 32)(100x^2 + 9)^{-9}.$$

    Next, we need to simplify this derivative by using algebra. Note that we can factor this. The way we do this is we pull out of each term that factor *with the smallest exponent*, which might be negative. For instance, in $x^{-5}y + x^{-4}z$, you would factor out an x^{-5}, since -5 is less than -4. This leaves you with $x^{-5}(y + xz)$. We get a single x in the second term because $-4 - -5 = 1$, or because $x^{-5} \cdot x = x^{-4}$, which is the factor with which we began.

    Using this reasoning, we will factor out a 20, $(100x^2 + 9)^{-10}$, and a $(5x^4 - 32x + 3)^4$ from both terms (in the process, we factor a 4 out of $20x^3 - 32$, which combines with the 5 to give us 20). This leaves:

    $$20(100x^2 + 9)^{-10}(5x^4 - 32x + 3)^4((-9)(5x^4 - 32x + 3)(10x) + (100x^2 + 9)(5x^3 - 8)).$$

    Now we expand and multiply to get:

    $$20(100x^2 + 9)^{-10}(5x^4 - 32x + 3)^4 \left(-450x^5 + 2880x^2 - 270x + 500x^5 + 45x^3 - 800x^2 - 72\right)$$

    By combining like terms, this becomes:

    $$20(100x^2 + 9)^{-10}(5x^4 - 32x + 3)^4 \left(50x^5 + 45x^3 + 2080x^2 - 270x - 72\right)$$

    We might try to factor the last factor, but it is a quintic (fifth degree polynomial). We do not expect you to factor this on the gateway exam.

22. Find the derivative of $g(x) = (2x + 1)^2(5x^8 + 2x)^3$.

SOLUTION: $(2x+1)^1(5x^8+2x)^2\left(2(130x^8+60x^7+10x+3)\right)$

23. Find the derivative of $g(x) = \frac{(5x^4+2x^2+e^x)^9}{(17x^2+x^{-1})^3}$.

SOLUTION: $(17x^2+x^{-1})^{-4}(5x^4+2x^2+e^x)^8(3060x^5-510x^4+408x^3+195x^2+42+153x^2e^x-102xe^x+9x^{-1}e^x+3x^{-2}e^x)$

24. Find the derivative of $g(x) = (x^4+1)^{-3}(x^5-2)^{-1}$.

SOLUTION: $(x^4+1)^{-4}(x^5-2)^{-2}\left(x^3(17x^5-5x+24\right)$

25. Find the derivative of $g(x) = (21x^{10}+11x^5+3)^5(x^4+x^2+1)^7$.

SOLUTION: $(21x^{10}+11x^5+3)^5(x^4+x^2+1)^7(x(1638x^{12}1344x^{10}+1050x^8+583x^7+429x^5+275x^3+84x^2+42)$

26. Find the derivative of $g(x) = \frac{(e^x+e^{x^2})^2}{(e^x+e^{-x})^3}$.

SOLUTION: $(e^x+e^{x^2})^2(e^x+e^{-x})^{-3}\left((4x-3)e^{x^2+x}+(4x+3)e^{x^2-x}-e^{2x}+5\right)$

27. Find the derivative of $g(x) = \frac{(x+1)^{99}}{(x-1)^{99}}$.

SOLUTION: $(x+1)^{98}(x-1)^{-100}\,(-198)$

28. Find the derivative of $g(x) = (5x^6+x^5+2)^7(5x^6+x^5+1)^{-7}$.

SOLUTION: $(5x^6+x^5+2)^6(5x^6+x^5+1)^{-8}\left(-7(30x^5+5x^4)\right)$

29. Find the derivative of $g(x) = (x^3+x^2+x+1)^3(e^{3x}+e^{2x}+e^x+1)^{-2}$.

SOLUTION: $(x^3+x^2+x+1)^2(e^{3x}+e^{2x}+e^x+1)^{-3}(3(x^3+9x^2+3x+2)e^{3x}+(2x^3+11x^2+8x+5)e^{2x}+(x^3+10x^2+7x+4)e^x+3(3x^2+2x+1))$

30. Find the derivative of $g(x) = (9x^9+6x^6+3x^3+1)^{-6}(x^2+1)^{-3}$.

SOLUTION: $(9x^9+6x^6+3x^3+1)^{-7}(x^2+1)^{-4}\left(-6x(90x^9+81x^7+42x^6+36x^4+12x^3+9x+1\right)$

31. Find $\frac{dy}{dx}$ if $x+y=xy^2$. You do not need to simplify your answer.

SOLUTION: $\frac{dy}{dx} = \frac{y^2-1}{1-2xy}$

32. Find $\frac{dy}{dx}$ if $xy+y=x$. You do not need to simplify your answer.

SOLUTION: $\frac{dy}{dx} = \frac{1-y}{x+1}$

33. Find $\frac{dy}{dx}$ if $x^2+xy^2=y$. You do not need to simplify your answer.

SOLUTION: $\frac{dy}{dx} = \frac{2x+y^2}{1-2xy}$

34. Find $\frac{dy}{dx}$ if $y + x^9 y = x + x^2 y$. You do not need to simplify your answer.

SOLUTION: $\frac{dy}{dx} = \frac{1+2xy-9x^8y}{1+x^9-x^2}$

35. Find $\frac{dy}{dx}$ if $x + x^2 + y = x^3 y^2$. You do not need to simplify your answer.

SOLUTION: $\frac{dy}{dx} = \frac{1+2x-3x^2y^2}{2x^3y^2-1}$

36. Find $\frac{dy}{dx}$ if $x^7 + x^8 y = xe^y$. You do not need to simplify your answer.

SOLUTION: $\frac{dy}{dx} = \frac{7x^6+8x^7y-e^y}{xe^y-x^8}$

37. Find $\frac{dy}{dx}$ if $e^x y = xe^y$. You do not need to simplify your answer.

SOLUTION: $\frac{dy}{dx} = \frac{e^xy-e^y}{xe^y-e^x}$

38. Find $\frac{dy}{dx}$ if $xy = \frac{x}{x^2+y}$. You do not need to simplify your answer.

SOLUTION: $\frac{dy}{dx} = \frac{1-3x^2y-y^2}{x^3+2xy}$

39. Find $\frac{dy}{dx}$ if $xy + xy^2 + x^2 y = 0$. You do not need to simplify your answer.

SOLUTION: $\frac{dy}{dx} = -\frac{1-3x^2y-y^2}{x^3+2xy}$

40. Find $\frac{dy}{dx}$ if $y + y^4 + 1 = x$. You do not need to simplify your answer.

SOLUTION: $\frac{dy}{dx} = \frac{1}{1+4y^3}$

41. Solve for x: $x^2 = \frac{x}{2}$.

SOLUTION: $x = 0$ or $x = \frac{1}{2}$.

42. Solve for x: $\frac{4^{x^3}}{6^{x/3}} = 1$.

SOLUTION: $x = 0$ or $x = \pm\sqrt{\frac{\ln 6}{3\ln 4}}$

43. Solve for x: $2^x - 6^{3x} = 0$.

SOLUTION: $x = 0$

44. Solve for x: $\frac{3^{2x}}{e^{x^2}} = 1$.

SOLUTION: $x = 0$ or $x = 2\ln 3$

45. Solve for x: $\log_2 x^2 - 2\log_2 \sqrt{x-2} = 3$.

SOLUTION: $x = 4$

46. Solve for x: $\ln 3x^3 - \ln \frac{1}{x^2} = 3$.

SOLUTION: $x = \sqrt[5]{\frac{e^3}{3}}$

47. Solve for x: $\frac{1}{3}\log_5 x^3 - \log_5(x+1) = -1$.

SOLUTION: $x = \frac{1}{4}$

48. Solve for x: $\log_6 x + \log_6(x+1) = 1$

SOLUTION: $x = 2$ ($x = -3$ is NOT a solution).

49. Solve for x: $\log_3 3x - \log_3 2x = 7$.

SOLUTION: No solution.

50. Solve for x: $\log_5 x = \log_7 2x$.

SOLUTION: $x = 2^{\frac{\ln 5}{\ln 7 - \ln 5}}$