

Name: _____

**Math Xb Midterm I Part II
Spring 2006**

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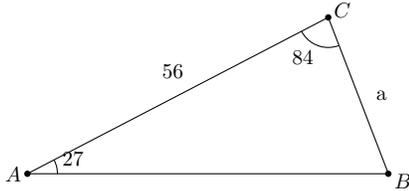
Problem Number	Possible Points	Score
1	10	
2	12	
3	12	
4	8	
5	12	
6	10	
7	10	
Total	74	

Directions—Please Read Carefully! You *are* allowed to use a calculator on this part of the exam, but no other aids are allowed. Be sure to write neatly—illegible answers will receive little or no credit.

To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit (except if the directions for that question specifically say no justification is necessary). **Write neatly**—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. **Good Luck!!!**

1. (10 points, 5 points each)

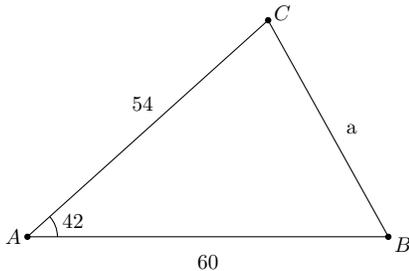
- (a) In the triangle below, some of the side lengths and angles (in *degrees*) have been marked. What is the length of the side labelled a ?



SOLUTION: We know two angles and a side, so the Law of Sines will help. First, note that $B = 180 - 84 - 27 = 69$. Therefore:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} \implies \frac{\sin(27)}{a} = \frac{\sin(69)}{56} \implies a = 27.23$$

- (b) In the triangle below, some of the side lengths and angles (in *degrees*) have been marked. What is the length of the side labelled a ?



SOLUTION: We know two sides and want to find the length of the side opposite a known angle. By the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc \cos A \implies a^2 = 54^2 + 60^2 - 2 \cdot 54 \cdot 60 \cos 42 \implies a^2 = 1700.42 \implies a = 41.24$$

2. (12 points, 3 points each) The cobra population in a jungle can be modeled with a sinusoidal function. The population is at its maximum of 2500 at time $t = 0$ months and first reaches its minimum of 1500 at time $t = 3$ months.

- (a) Sketch a graph of the snake population over 12 months ($0 \leq t \leq 12$). Label the maximum and minimum points on the graph and sketch and label the balance line.

- (b) Write a function $P(t)$ for the graph you sketched in part (a).

SOLUTION: The population is at its maximum at time 0, so we use cosine as the basis of our function, i.e. $P(t) = A \cos(Bt) + D$. The amplitude is half the distance between the max of 2500 and min of 1500, so $A = 500$. The balance value is the average of the max and min, so $D = 2000$. Finally the period p is twice the time between the max and next min, so $B = \frac{2\pi}{p} = \frac{2\pi}{6} = \frac{\pi}{3}$. Therefore, $P(t) = 500 \cos(\frac{\pi}{3}t) + 2000$.

- (c) Find the first time t when the cobra population is 2250.

SOLUTION: We set $P(t) = 2250$ and solve for t .

$$\begin{aligned} 2250 &= 500 \cos\left(\frac{\pi}{3}t\right) + 2000 \\ 250 &= 500 \cos\left(\frac{\pi}{3}t\right) \\ \frac{1}{2} &= \cos\left(\frac{\pi}{3}t\right) \\ \frac{\pi}{3}t &= \frac{\pi}{3} + 2\pi n \text{ or } \frac{\pi}{3}t = \frac{5\pi}{3} + 2\pi n \\ t &= 1 + 6n \text{ or } t = 5 + 6n. \end{aligned}$$

So the first time after time 0 when the population is 2250 is at $t = 1$ month.

- (d) Find ALL other times t when the cobra population is 2250 during the first year.

SOLUTION: By part c, the solutions with $0 \leq t \leq 12$ are $t = 1, 5, 7,$ and 11 months. This part could also be done by using the solution $t = 1$ and noting the symmetry and periodicity in the graph in part (a).

3. (12 points, 4 points each)

- (a) I have a set of 5 Russian dolls, which fit inside one another. If I put the dolls in order of size, each is exactly $\frac{4}{5}$ the size of the previous doll. The largest doll is half a yard high. If I lay the dolls head-to-toe, how far do they stretch in total?

SOLUTION: Here $a = \frac{1}{2}$, $r = \frac{4}{5}$, and $n = 4$ (so $n + 1 = 5$).

$$\frac{1}{2} + \frac{1}{2} \left(\frac{4}{5}\right) + \frac{1}{2} \left(\frac{4}{5}\right)^2 + \frac{1}{2} \left(\frac{4}{5}\right)^3 + \frac{1}{2} \left(\frac{4}{5}\right)^4 = \left(\frac{1}{2} - \frac{1}{2} \left(\frac{4}{5}\right)^5\right) / \left(1 - \frac{4}{5}\right) \approx 1.680800 \text{ yards.}$$

- (b) Now suppose I have a nesting set of 50 Russian dolls, each doll $\frac{4}{5}$ the size of the previous doll. The largest doll is half a yard high. If I lay the dolls head-to-toe, how far do they stretch in total? Give six decimal places in your answer.

SOLUTION: Here $a = \frac{1}{2}$, $r = \frac{4}{5}$, and $n = 49$ (so $n + 1 = 50$).

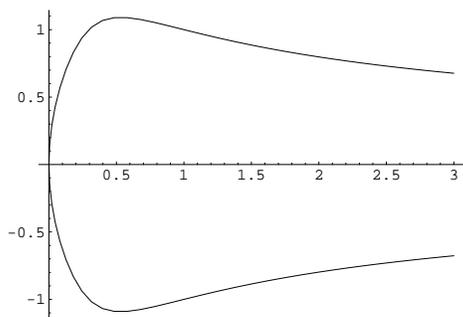
$$\frac{1}{2} + \frac{1}{2} \left(\frac{4}{5}\right) + \frac{1}{2} \left(\frac{4}{5}\right)^2 + \dots + \frac{1}{2} \left(\frac{4}{5}\right)^{49} = \left(\frac{1}{2} - \frac{1}{2} \left(\frac{4}{5}\right)^{50}\right) / \left(1 - \frac{4}{5}\right) \approx 2.499964 \text{ yards.}$$

- (c) Now suppose that I have a nesting set of infinitely many dolls, each doll $\frac{4}{5}$ the size of the previous doll with the largest doll half a yard high. If the dolls are all lying head-to-toe, how far do they stretch in total (note that the answer here might be infinitely far or the answer might be a finite number – remember to justify whatever answer you give.)

SOLUTION: Here $a = \frac{1}{2}$ and $r = \frac{4}{5}$. Since $|r| < 1$ the answer will be finite.

$$\frac{1}{2} + \frac{1}{2} \left(\frac{4}{5}\right) + \frac{1}{2} \left(\frac{4}{5}\right)^2 + \frac{1}{2} \left(\frac{4}{5}\right)^3 + \frac{1}{2} \left(\frac{4}{5}\right)^4 + \dots = \frac{\frac{1}{2}}{1 - \frac{4}{5}} = 2.5 \text{ yards.}$$

4. (8 points) Find the equation of the tangent line to $(x^2y^3 + y)^2 = 4x$ at the point $(1, 1)$



SOLUTION: We use implicit differentiation to find the slope of the tangent line at $(1, 1)$.

$$\begin{aligned}(x^2y^3 + y)^2 &= 4x \\ \frac{d}{dx} ((x^2y^3 + y)^2) &= \frac{d}{dx} (4x) \\ 2(x^2y^3 + y) \frac{d}{dx} (x^2y^3 + y) &= 4 \\ 2(x^2y^3 + y)(2xy^3 + 3y^2 \frac{dy}{dx} x^2 + \frac{dy}{dx}) &= 4\end{aligned}$$

Plugging in $x = 1$ and $y = 1$ gives:

$$\begin{aligned}2(1 + 1)(2 + 3 \frac{dy}{dx} + \frac{dy}{dx}) &= 4 \\ 4(2 + 3 \frac{dy}{dx} + \frac{dy}{dx}) &= 4 \\ 2 + 3 \frac{dy}{dx} + \frac{dy}{dx} &= 1 \\ 4 \frac{dy}{dx} &= -1 \\ \frac{dy}{dx} &= -\frac{1}{4}\end{aligned}$$

The line with slope $m = -\frac{1}{4}$ through the point $(1, 1)$ has the equation $y - 1 = -\frac{1}{4}(x - 1)$ or $y = -\frac{1}{4}x + \frac{5}{4}$, which seems reasonable given the picture.

5. (12 points, 4 points each)

(a) Find

$$\cos(\alpha + \beta) - \cos(\alpha - \beta)$$

in terms of $\sin \alpha$, $\cos \alpha$, $\sin \beta$, and $\cos \beta$ (your answer may involve some or all of these four quantities).

SOLUTION: Using the addition formula for cosine, that cosine is even, and that sine is odd, we have:

$$\begin{aligned}\cos(\alpha + \beta) - \cos(\alpha - \beta) &= (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) - (\cos(\alpha)\cos(-\beta) - \sin(\alpha)\sin(-\beta)) \\ &= (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) - (\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)) \\ &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) - \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ &= -2\sin(\alpha)\sin(\beta)\end{aligned}$$

(b) Find an expression for

$$(\sin x + \cos x)^2$$

in terms of $\sin 2x$.

SOLUTION:

$$\begin{aligned}(\sin x + \cos x)^2 &= \sin^2 x + \cos^2 x + 2\sin x \cos x \\ &= 1 + \sin(2x)\end{aligned}$$

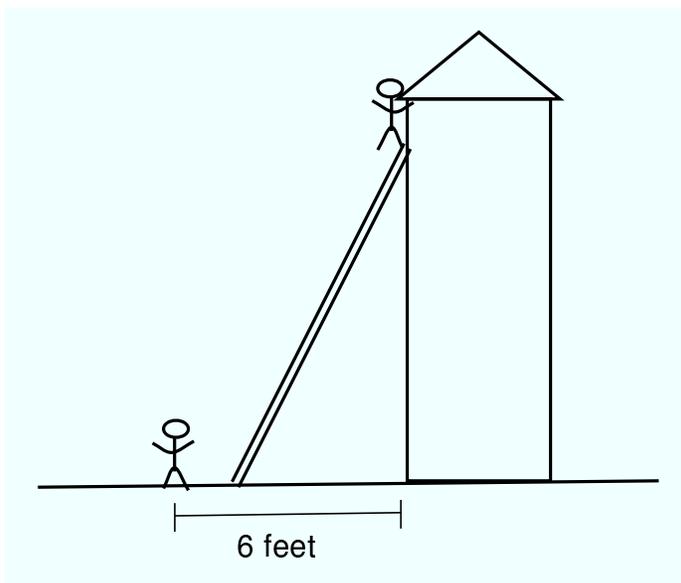
(c) Use trig identities and the trig values you know to find the *exact* value of $\sin 75^\circ$. (Note that you must show how you have used trig identities to receive credit and that a decimal answer will not get you any credit.)

SOLUTION: We express 75 as the sum of angles whose sines we know from familiar triangles on the unit circle (or memory) and use the angle addition formula for sine:

$$\begin{aligned}\sin 75 &= \sin(30 + 45) \\ &= \sin 30 \cos 45 + \sin 45 \cos 30 \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

6. (10 points) Gladys Kravitz is convinced that her neighbor is a witch and this time she's going to get proof. She takes a 10-foot ladder over to her neighbor's house where she plans to look through a window and take the photo which will prove that she's not crazy and that her neighbor is really a witch.

Gladys is perched precariously at the top of a 10-foot ladder leaning against the back wall of her neighbor's house. Her husband, Abner, sees what she is up to and comes over to try to stop her. The ladder starts to slide down the wall. Abner is standing on the ground 6 feet away from the wall (behind Gladys). When the base of the ladder hits Abner, the top of the ladder is sliding down the wall at a rate of 4 feet per second. How fast is the base of the ladder moving when it hits Abner?



SOLUTION: Let y be the height of the top of the ladder and let x be the distance from the bottom of the ladder to the wall (these are really both functions of time). Then at all times we have:

$$x^2 + y^2 = 10^2$$

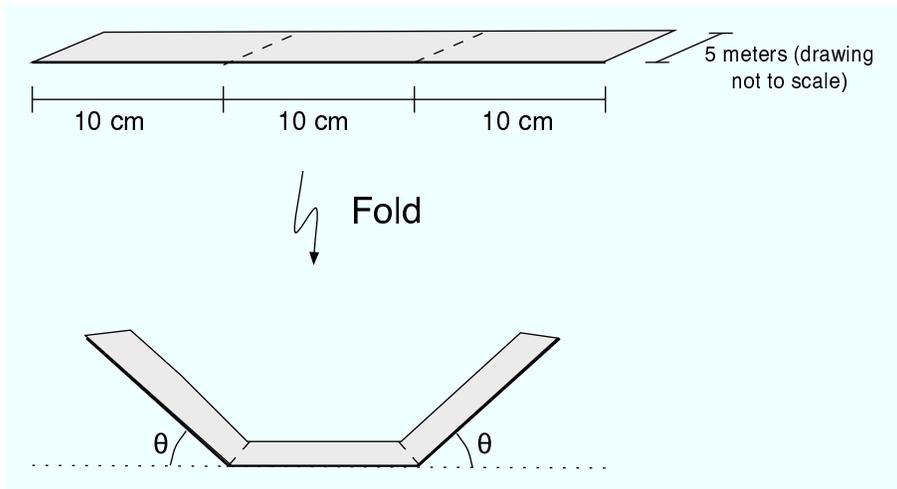
Differentiating with respect to time gives:

$$\begin{aligned} 2x \frac{dx}{dt} + 2y \frac{dy}{dt} &= 0 \\ \frac{dx}{dt} &= -\frac{y}{x} \frac{dy}{dt} \end{aligned}$$

Plugging in $\frac{dy}{dt} = -4$, $x = 6$, and $y = \sqrt{10^2 - 6^2} = 8$ gives:

$$\frac{dx}{dt} = -\frac{8}{6}(-4) = \frac{16}{3} \text{ ft/sec.}$$

7. (10 points) A rain gutter is to be constructed from a metal sheet of width 30 centimeters and length 5 meters by bending up one-third of the sheet on each side through an angle θ as shown below. How should θ be chosen so that the gutter will carry the maximum amount of water? *You must carefully justify your work to get full credit. In particular, you need to show that you have found a maximum.*



SOLUTION: Since the length of the gutter is fixed at five meters, it is sufficient to find the θ that maximized the cross-sectional area of the gutter, which has the shape of a trapezoid. We must first express this area A as a function of the angle θ . We can break the trapezoid into two triangles of equal area and a rectangle. The height h of the triangles and the rectangle is $10 \sin \theta$. The width w of the rectangle is 10. The base width b of the triangles is $10 \cos \theta$. Therefore:

$$A(\theta) = lh + 2\left(\frac{1}{2}bh\right) = 100 \sin \theta + 100 \sin \theta \cos \theta$$

It is clear from the diagram that the maximum area occurs when $0 \leq \theta \leq \frac{\pi}{2}$. The values at the endpoints are $A(0) = 0$ and $A\left(\frac{\pi}{2}\right) = 100$. We check for local extrema in the interior by setting the derivative equal to 0:

$$\begin{aligned} A'(\theta) &= 100 \cos \theta + 100(-\sin^2 \theta + \cos^2 \theta) = 0 \\ \cos \theta - (1 - \cos^2 \theta) + \cos^2 \theta &= 0 \\ 2 \cos^2 \theta + \cos \theta - 1 &= 0 \\ (2 \cos \theta - 1)(\cos \theta + 1) &= 0 \\ \cos \theta &= \frac{1}{2} \text{ or } \cos \theta = 0 \end{aligned}$$

There are no solutions to $\cos \theta = 0$ with $0 \leq \theta \leq \frac{\pi}{2}$. The solution to $\cos \theta = \frac{1}{2}$ in this range is $\theta = \frac{\pi}{3}$. Therefore the only interior extrema occurs at $\theta = \frac{\pi}{3}$ and $A\left(\frac{\pi}{3}\right) = 75\sqrt{3} \approx 129.9 > 100$. Therefore $\theta = \frac{\pi}{3}$ is the angle which maximizes the volume of the gutter.