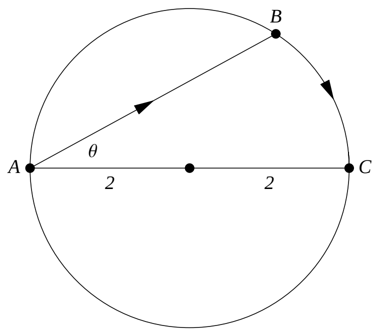
Math Xb
Applications of Trig Derivatives

1. A woman at point A on the shore of a circular lake with radius 2 miles wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time. She can walk at the rate of 4 miles per hour and row a boat at 2 miles per hour. How should she proceed?



LAW OF COSINES SOLUTION: We are trying to minimize time, and we can use the equation "distance equals rate times time," or $d = rt$. So let d_w, r_w, and t_w be the distance, rate, and time (respectively) for her walking and d_r, r_r, and t_r be the distance, rate, and time (respectively) for her rowing. The problem gives us that $r_r = 2$ mph and $r_w = 4$ mph. Then her total time is

$T(d_r, d_w) = \frac{d_r}{2} + \frac{d_w}{4}$.

This is not ideal, though, because there are two variables (d_r and d_w). Let's try to write these in terms of one variable.

If we label X as the center of the circle in the drawing, we can consider the triangle ABX. Using standard conventions, we denote the length of side AB by "x," the length of BX by "a", and the length of AX by "b." We would like to find the length of side AB, since that length is d_r. We will use trigonometry to do this. Side AX and XB both have length 2, since these are radii of the circle. So $b = 2$ and $a = 2$. Therefore, this is an isosceles triangle, and we know that angle ABX is equal to θ, since base angles of isosceles triangles are congruent (i.e. "the same"). Since the angles of a triangle sum to π radians (equivalently, 180 degrees), we see that angle AXB must have measure $\pi - 2\theta$.

Now we can apply the law of cosines to get $2^2 = x^2 + 2^2 - 2(2)x\cos\theta$. Then we can cancel the 2^2s on each side to get:

$0 = x^2 - 4x\cos\theta = x(x - 4\cos\theta)$.

From this, we conclude that $x = 0$ or $x = 4\cos\theta$. Since x is the length of a side of a triangle, it cannot be zero. So we are left with $x = 4\cos\theta$, or equivalently $d_r = 4\cos\theta$.

We can find the distance that she walks if we know the angle BXC. The formula for the length of an arc of a circle is "radius times angle," where the angle must be in radians. Since angle AXB has measure $\pi - 2\theta$ and AXC is a straight line (so pi radians), we know that angle BXC is 2θ. Then "radius times angle" becomes "$2 \times 2\theta = 4\theta$. So she walks $d_w = 4\theta$ miles.

So her total time is $\frac{d_r}{2} + \frac{d_w}{4} = \frac{4\cos\theta}{2} + \frac{4\theta}{4} = 2\cos\theta + \theta$. This yields the one-variable function:

$T(\theta) = 2\cos\theta + \theta$.

We want to minimize this function, and we know that local minima occur when the derivative equals zero. Then consider

$T'(\theta) = -2\sin\theta + 1$.

We set this equal to zero to get $0 = -2\sin\theta + 1$, or $\sin\theta = \frac{1}{2}$.

Before solving this trig equation, we first consider what θ can be. This picture only makes sense for θ's where $0 \le \theta \le \frac{\pi}{2}$, where $\theta = 0$ corresponds to her rowing the whole way, $\theta = \frac{\pi}{2}$ corresponds to her walking the whole way, and all values between correspond to one angle of a triangle.

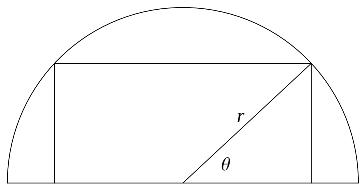
With this in mind, we know that the only solution for this situation to the equation $\sin\theta = \frac{1}{2}$ is $\theta = \frac{\pi}{6}$.

HOWEVER, we still need to test to make sure that this is a minimum, since we are trying to minimize time. The first or second derivative test might be useful here, but I am going to test it by using the fact that θ lies in the *closed* interval $[0, \frac{\pi}{2}]$. Since it is a closed interval, we know that both a min and a max must occur, and they occur at either a critical point (e.g. $\frac{\pi}{6}$) or an endpoint (e.g. 0 and $\frac{\pi}{2}$). So we test these by plugging them back into the *original* function, since we want to actually find out which angle causes the most travel time and which angle causes the least travel time. Then:

$T(0) = 2\cos 0 + 0 = 2$
$T(\frac{\pi}{6}) = 2\cos\frac{\pi}{6} + \frac{\pi}{6} = \sqrt{3} + \frac{\pi}{6} \approx 2.3$
$T(\frac{\pi}{2}) = 2\cos\frac{\pi}{2} + \frac{\pi}{2} = \frac{\pi}{2} \approx 1.57$

So the maximum travel time is about 2.3 hours and occurs at the critical point $\frac{\pi}{6}$, and the least amount of travel time is about 1.57 hours and occurs at the end point $\frac{\pi}{2}$. So, to minimize time, she should walk the entire way.

2. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r. (Hint: Write the area of the rectangle as a function of θ.)



SOLUTION: Consider the right triangle in the picture that has hypotenuse r (it is the only right triangle that is drawn in the picture). Using SOH-CAH-TOA reasoning, we know that the vertical side of this right triangle has length $r\sin\theta$ and the horizontal side has length $r\cos\theta$.

Now we can see that the rectangle shares the vertical leg of the right triangle, so one of the sides of the rectangle has length $r\sin\theta$. The other side of the rectangle is twice the horizontal leg of the right triangle, or $2r\cos\theta$. Therefore, we can find the area of the rectangle in terms of θ:

$A(\theta) = (2r\cos\theta)(r\sin\theta) = r^2(2\sin\theta\cos\theta)$

Now, we can use the trig identity $\sin(2\theta) = 2\sin\theta\cos\theta$ to get:

$A(\theta) = r^2\sin(2\theta)$

To maximize, we take the derivative (using the chain rule and realizing that r is a constant) and set it equal to zero:

$A'(\theta) = r^2(\cos(2\theta))(2) = 2r^2\cos(2\theta)$.

Setting this equal to zero, we get:

$0 = 2r^2\cos(2\theta)$.
$0 = \cos(2\theta)$

Therefore, $2\theta = \frac{\pi}{2}n$ for any integer n, so $\theta = \frac{\pi}{4}n$. However, θ is an angle of a triangle, so it must lie between 0 and $\frac{pi}{2}$, so the only solution is $\theta = \frac{\pi}{4}$.

Finally, we need to see if this is a maximum. The second derivative is easy to find, so I will use the second derivative test:

$A''(\theta) = -4r^2\sin(2\theta)$.
$A''(\frac{\pi}{4}) = -4r^2\sin(2\frac{\pi}{4}) = -4r^2\sin\frac{pi}{2} = -4r^2 < 0$.

Since we got a negative number, the second derivative tells us that $\theta = \frac{\pi}{4}$ yields a maximum area. To finish the problem, we just need to compute what this area is:

$A(\frac{\pi}{4}) = r^2\sin(2\frac{\pi}{4}) = r^2\sin(\frac{\pi}{2}) = r^2$.

So the maximum area is r^2, and we are done.

3. Let $f(\theta) = \sin^2(\theta + \pi/4)$, for $\frac{pi}{2} \geq a > 0$ constant and $0 \leq \theta \leq 2\pi$.

   (a) Find the intervals on which f increases and decreases.

   (b) Find the local maximum and minimum values of f.

   (c) Find the intervals on which f is concave up and concave down.

   (d) Find the inflection points of f.

   (e) Use your answers to parts (a–d) to sketch the graph of f. Compare this with what you get using your graphing calculator.

SOLUTION:

(a) The function f increases when its derivative is positive, and decreases when the derivative is negative. $f'(\theta) = 2\sin(\theta + a)\cos(\theta + a) = \sin(2(\theta + a))$. So we just need to figure out where the derivative $\sin(2(\theta + a))$ is positive. But we know that $\sin x$ is positive on the intervals

$$...(-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup (4\pi, 5\pi) \cup (6\pi, 7\pi)...$$

and $\sin x$ is negative on the intervals

$$...(-3\pi, -2\pi) \cup (-\pi, 0) \cup (\pi, 2\pi) \cup (3\pi, 4\pi) \cup (5\pi, 6\pi)...$$

Shall we examine one of those intervals? Let's shall. Let's look at $(2\pi, 3\pi)$ from the positive group of intervals. Then $\sin x$ is positive if $2\pi < x < 3\pi$. This means that $\sin(2\theta + a)$ is positive if

$$2\pi < 2(\theta + a) < 3\pi,$$
$$\pi < \theta + a < \frac{3\pi}{2}$$
$$\pi - a < \theta < \frac{3\pi}{2} - a$$

Since $0 < a < \frac{\pi}{2}$, both $\pi - a$ and $\frac{3\pi}{2} - a$ are still in the interval $[0, 2\pi]$. So a partial answer to the question is that $f(\theta)$ is increasing on the interval $(\pi - a, \frac{3\pi}{2} - a)$.

We can do similar reasoning on the interval $(0, \pi)$ to get

$$-a < \theta < \frac{\pi}{2} - a.$$

This time, we see that much of the interval $(-a, \frac{\pi}{2} - a)$ is NOT in the interval of allowable θ's $(0 \leq \theta \leq 2\pi)$. This time, we get a partial answer of $(0, \frac{\pi}{2} - a)$, since we need to lop off the non-positve values.

To repeat, $\sin x$ is positive on

$$...(-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup (4\pi, 5\pi) \cup (6\pi, 7\pi)...$$

and negative on the intervals

$$...(-3\pi, -2\pi) \cup (-\pi, 0) \cup (\pi, 2\pi) \cup (3\pi, 4\pi) \cup (5\pi, 6\pi)...$$

Using reasoning from above to each of the intervals individually, we get that $\sin(2(\theta + a))$ is positive on

$$...(-\pi - a, -\frac{\pi}{2} - a) \cup (-a, \frac{\pi}{2} - a) \cup (\pi - a, \frac{3\pi}{2} - a) \cup (2\pi - a, \frac{5\pi}{2} - a) \cup (3\pi - a, \frac{7\pi}{2} - a)...$$

and negative on the intervals

$$...(-\frac{3\pi}{2} - a, -\pi - a) \cup (-\frac{\pi}{2} - a, -a) \cup (\frac{\pi}{2} - a, \pi - a) \cup (\frac{3\pi}{2} - a, 2\pi - a) \cup (\frac{5\pi}{2} - a, 3\pi - a)...$$

Since some of these intervals lie either partially or completely out of the interval that we are interested in ($[0, 2\pi]$), we can reduce these down to$\sin(2(\theta + a))$ is positive on

$$(0, \frac{\pi}{2} - a) \cup (\pi - a, \frac{3\pi}{2} - a) \cup (2\pi - a, 2\pi)$$

and negative on the intervals

$(\frac{\pi}{2} - a, \pi - a) \cup (\frac{3\pi}{2} - a, 2\pi - a)$

This ends the first part of this question.

(b) We can find local extrema by taking derivatives and finding critical points. Recall that $f'(\theta) = \sin(2(\theta + a))$, so we want to find out where $\sin(2(\theta + a)) = 0$.

We know that $\sin x = 0$ when $x = n\pi$ for any integer n. Therefore, $\sin(2(\theta + a)) = 0$ when $2(\theta + a) = n\pi$. We can solve for θ:

$2(\theta + a) = n\pi$
$\theta + a = \frac{\pi}{2}n$
$\theta = \frac{\pi}{2}n - a$

Now we just need to figure out which of these solutions lie in $[0, 2\pi]$. If $n \leq 0$, then θ is negative, so we won't consider it. If $n = 1$, we get $\theta = \frac{\pi}{2} - a$, which is in $[0, 2\pi]$ (since $0 < a < \frac{\pi}{2}$). We continue to check until we get to $n = 4$, which gives $\theta = 2\pi - a$. At $n = 5$, we get $\theta = \frac{5\pi}{2} - a$, which is larger than 2π (again, since $0 < a < \frac{\pi}{2}$). Therefore, we get the following four critical points (corresponding to $n = 1, 2, 3, 4$):

$\theta = \frac{\pi}{2} - a, \pi - a, \frac{3\pi}{2} - a, 2\pi - a.$

We can use the second derivative test to figure out which are maxima and which are minima. $f''(\theta) = 2\cos(2(\theta + a))$. Plugging in $\frac{\pi}{2} - a$ or $\frac{3\pi}{2} - a$, we get that the cosine is negative, so these values correspond to maxima. Similarly, the two other values cause the cosine to go positive, so they are minima.

(c) We use the exact same process for the intervals where f is increasing and decreasing to find out where it is concave up and down, except that we will use the reasoning on $f''(\theta) = 2\cos(2(\theta + a))$ instead of $f'(\theta) = \sin(2(\theta + a))$. This gives us answers of:

Concave up: $(0, \frac{\pi}{4} - a) \cup (\frac{3\pi}{4} - a, \frac{5\pi}{4} - a) \cup (\frac{7\pi}{4} - a, 2\pi)$ if $0 < a < \frac{\pi}{4}$
$(\frac{3\pi}{4} - a, \frac{5\pi}{4} - a) \cup (\frac{7\pi}{4} - a, \frac{9\pi}{4} - a)$ if $\frac{\pi}{4} \leq a < \frac{\pi}{2}$

In each case, the function is concave down where it isn't concave up (it is getting late, so I am getting lazy).

(d) The inflection points where the function changes from concave up to concave down (or vice versa). We can see from the previous problem that this occurs at the endpoints of the intervals. So the inflection points are where

$\theta = \frac{\pi}{4} - a$ (only if $0 < a < \frac{\pi}{4}$), $\frac{3\pi}{4} - a$, $\frac{5\pi}{4} - a$, $\frac{7\pi}{4} - a$, $\frac{9\pi}{4} - a$ (this one only if $\frac{\pi}{4} \leq a < \frac{\pi}{2}$.

(e) Check you sketch of the graph by using a graphing calculator. I need to go celebrate St. Patrick's Day.

4. Follow the instructions in the previous question for $f(t) = t + \cos t, \quad -2\pi \leq t \leq 2\pi$.

SOLUTION: The reasoning will be exactly like problem 3.

(a) $f'(t) = 1 - \sin t$, which is always positive or zero (since $\sin t \leq 1$). So the function is always increasing.

(b) To have a local min, you have to have the function decrease for a while, and then increase. Local max's are similar. Since this function is always increasing, it has no local min's nor local max's.

(c) $f''(t) = -\cos t$. This is positive on $(-\frac{3\pi}{2}, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2})$, so this is where $f(t)$ is concave up. It is concave down everywhere else on $[-2\pi, 2\pi]$.

(d) The inflection points are again where it changes from concave up to down (or vice versa), and these are exactly the endpoints of the intervals in the previous question. So the inflection points are $t = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$.

(e) Again, you can check the graph on a graphing calculator.