Name:______________________________

Math Xb Midterm II Part 1
Spring 2006

Please circle your section:

Thomas Barnet-Lamb	Jon Bloom	Angela Vierling-Claassen
MWF 10–11	**MWF 10–11**	MWF 10–11
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Part One Scores

Problem Number	Possible Points	Score
1	9	
2	6	
3	12	
4	8	
5	7	
Total	42	

Midterm II Scores

	Possible Points	Score
Part One	42	
Part Two	58	
Total	100	

Directions—Please read carefully!

You *are not allowed* to use a calculator or any other aids on this part of the exam. When you are finished with this part of the exam, you may turn it in to the proctor. Once you have turned in this first part of the exam, you may not look at it again, so be sure you have finished it completely before turning it in. **You may not use a calculator until you have turned in this first part of the exam.**

To receive full credit on a problem, you will need to justify your answers carefully—**unsubstantiated answers will receive little or no credit** (except if the directions for that question specifically say no justification is necessary). Be sure to **write neatly—illegible answers will receive little or no credit**. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers.

Good luck!

1. (9 points, 3 points each) Find each of the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{e^{3x}+x}{e^x+3}$

SOLUTION: Since the denominator is continuous, we may simply plug $x=0$ into the denominator to determine that its limit is 4. Similarly, the numerator is continuous, so we may simply plug $x=0$ into the numerator to determine that its limit is 1. Therefore, the limit is $\frac{1}{4}$.

(b) $\displaystyle\lim_{x\to 0} \frac{e^{3x}+x-1}{e^x-1}$

SOLUTION: The limits of the numerator and the denominator are both 0, so this is an indeterminate form. Therefore, we need to use L'Hospital's Rule to get that this limit is equal to the following limit (as long as this new limit exists):

$$\lim_{x\to 0} \frac{e^{3x}+x-1}{e^x-1} = \lim_{x\to 0} \frac{3e^{3x}+1}{e^x} = \frac{4}{1} = 4.$$

(c) $\displaystyle\lim_{x\to\infty} \frac{x^2}{1.001^x}$

SOLUTION: The limits of the numerator and the denominator are both ∞, so this is an indeterminate form. Therefore, we need to use L'Hospital's Rule to get that this limit is equal to the following limit (as long as this new limit exists):

$$\lim_{x\to\infty} \frac{x^2}{1.001^x} = \lim_{x\to\infty} \frac{2x}{(\ln 1.001)1.001^x}$$

This new limit is still indeterminate of the form "$\frac{\infty}{\infty}$," so we apply L'Hospital's Rule again:

$$\lim_{x\to\infty} \frac{x^2}{1.001^x} = \lim_{x\to\infty} \frac{2x}{(\ln 1.001)1.001^x} = \lim_{x\to\infty} \frac{2}{(\ln 1.001)^2 1.001^x}$$

This newest limit goes to "$\frac{2}{\infty}$," so the limit is 0. Therefore, the final answer is 0 by L'Hospital's Rule.

2. (6 points, 3 points each) Compute the following derivatives.

(a) $f'(x)$, where $f(x) = \arcsin(x)(x^2 - 1)$

SOLUTION: $f'(x) = \arcsin(x)(x^2-1)' + (\arcsin(x))'(x^2-1) = \arcsin(x)(2x) + \frac{1}{\sqrt{1-x^2}}(x^2-1)$.

(b) $h'(x)$, where $h(x) = \arctan(\arccos(x))$

We will use the chain rule: $h(x) = f(g(x))$, where $f(x) = \arctan x$ and $g(x) = \arccos x$. Then $h'(x) = f'(g(x))g'(x)$. So we get:

$$h'(x) = f'(g(x))g'(x) = \left(\frac{1}{1 + (\arccos x)^2}\right)\left(\frac{1}{-\sqrt{1-x^2}}\right).$$

3. (12 points, 3 points each) Find the following definite and indefinite integrals.

(a) $\int [(x^6+1)^9 + (x^6+1)^8 + (x^6+1)^7](6x^5)dx$

SOLUTION: Let $u = x^6 + 1$, then $du = 6x^5$. Then our new integral is:

$$\int [u^9 + u^8 + u^7]\; du = \frac{u^{10}}{10} + \frac{u^9}{9} + \frac{u^8}{8} + C = \frac{(x^6+1)^{10}}{10} + \frac{(x^6+1)^9}{9} + \frac{(x^6+1)^8}{8} + C$$

(b) $\int \frac{\arcsin x}{\sqrt{1-x^2}} dx$

SOLUTION: Let $u = \arcsin x$. Then $du = \frac{1}{\sqrt{1-x^2}}$. Then our new integral is:

$$\int u\; du = \frac{u^2}{2} + C = \frac{(\arcsin x)^2}{2} + C.$$

(c) $\int \left(\frac{10}{x} + 1 + e^x\right) dx$

SOLUTION: We know all three of these antiderivatives immediately, so the integral is equal to $10 \ln |x| + x + e^x + C$.

(d) $\int_0^5 \left(5\sin x + \sqrt{x}\right) dx$

SOLUTION: $\int_0^5 \left(5\sin x + \sqrt{x}\right) dx = -5\cos x + \frac{2}{3}x^{3/2}\Big|_0^5 =$

$= -5\cos 5 + \frac{2}{3}(5)^{3/2} - (-5\cos 0 + 0) = -5\cos 5 + \frac{2}{3}(5)^{3/2} - 5$

4. (8 points) Here is a list of expressions involving integrals. Some pairs of expressions are equal. Pair up any equal expressions below. You don't have to show your work.

(a) $2\int_0^3 \arctan(x)dx + \int_3^6 2\arctan(x)dx$

(b) $\int_0^{1/2} (\arctan(x) - \arcsin(x))\, dx + \int_{1/2}^1 \arctan(x)dx$

(c) $\int_0^{10} \arctan(x)dx$

(d) $\int_0^3 \arctan(x)dx$

(e) $2\int_0^5 \arctan(x)dx + 2\int_5^6 \arctan(x)dx$

(f) $\int_0^{1/2} (\arctan(x) + \arcsin(x))\, dx + \int_{1/2}^1 \arctan(x)dx$

(g) $\frac{1}{2}\int_0^{10} 2\arctan(x)dx$

(h) $\int_0^2 \arctan(x)dx - \int_3^2 \arctan(x)dx$

(i) $\int_0^1 \arctan(x)dx + \int_{1/2}^0 \arcsin(x)dx$

SOLUTION:

(a) is equal to (e)
(b) is equal to (i)
(c) is equal to (g)
(d) is equal to (h)

5. (7 points) Put the following in *ascending* order in the spaces provided below. You do not need to justify your solution. [*Hint*: Think about which expressions are positive, which are negative, and which are zero. A picture may be helpful.]

(a) $\int_4^9 \ln t \, dt$

(b) $\ln 4 + \ln 5 + \ln 6 + \ln 7 + \ln 8$

(c) $\ln 5 + \ln 6 + \ln 7 + \ln 8 + \ln 9$

(d) Zero

(e) $\sum_{k=0}^{9} \frac{\ln(4 + \frac{k}{2})}{2}$

(f) $\ln(4/9)$

(g) $\lim_{h \to 0} \frac{\ln(4+h) - \ln 4}{h}$

SOLUTION: The first expression is the exact area of $\ln t$ between $t = 4$ and $t = 9$, the second expression is a left-hand underestimate of that area using four rectangles, the third is a right-hand overestimate of the first area, the fifth is a better (bigger) left-hand understimate of the first area using eight rectangles, and the sixth is a negative number. The seventh is the derivative of $\ln t$ evaluated at $t = 4$, which is $\frac{1}{t}$, so the value is $\frac{1}{4}$, which is smaller than any of the areas.

So the final answer is:

$$\ln(4/9) \leq 0 \leq \lim_{h \to 0} \frac{\ln(4+h) - \ln 4}{h} = \frac{1}{4} \leq \ln 4 + \ln 5 + \ln 6 + \ln 7 + \ln 8 \leq \sum_{k=0}^{9} \frac{\ln(4 + \frac{k}{2})}{2} \leq \int_4^9 \ln t \, dt \leq \ln 5 + \ln 6 + \ln 7 + \ln 8 + \ln 9.$$

So the answer is:

$f \leq d \leq g \leq b \leq e \leq a \leq c.$

Name:________________________________

Math Xb Midterm II Part 2
Spring 2006

Please circle your section:

Thomas Barnet-Lamb MWF 10–11	Jon Bloom **MWF 10–11**	Angela Vierling-Claassen MWF 10–11
Bret Benesh MWF 11-12	Jon Bloom **MWF 11-12**	

Part Two Scores

Problem Number	Possible Points	Score
1	12	
2	10	
3	12	
4	12	
5	12	
Total	58	

Directions—Please Read Carefully! You *are* allowed to use a calculator on this part of the exam, but no other aids are allowed. **You cannot start to use your calculator until you have turned in the first part of the exam.** Be sure to write neatly—illegible answers will receive little or no credit.

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1. (12 points) A ball is thrown up in the air from an initial height of 0 ft and with an initial velocity of 64 ft/sec. It travels straight up and straight down to its original position. The acceleration due to gravity is -32 ft/sec^2. Therefore the velocity at time t is given by $v(t) = -32t + 64$.

   (a) What is the average velocity of the ball between $t = 0$ and $t = 3$?

   SOLUTION: The definition of average value for a function on the interval $[a, b]$ is $\frac{\int_a^b f(x)\, dx}{b-a}$. Therefore, the average velocity of the ball between $t = 0$ and $t = 1$ is:

   $$\frac{\int_0^3 v(t)\, dt}{3-0} = \frac{\int_0^3 -32t + 64\, dt}{3} = \frac{-16t^2 + 64t\Big|_0^3}{3} = \frac{(-16(3)^2 + 64(3)) - (-16(0)^2 + 64(0))}{3} = \frac{48}{3} \text{ feet/second.}$$

   (b) What is the average speed of the ball between $t = 0$ and $t = 3$?

   SOLUTION: The speed is the absolute value of the velocity. So the speed equals the velocity on the interval $[0, 2]$ (since $v(t)$ is nonnegative there), and the speed is the opposite of the velocity on the interval $(2, 3]$ (since $v(t)$ is negative there).

   So we have:

   $$\frac{\int_0^3 |v(t)|\, dt}{3-0} = \frac{\int_0^2 v(t)\, dt + \int_2^3 -v(t)\, dt}{3} = \frac{\int_0^3 -32t + 64\, dt + \int_2^3 32t - 64\, dt}{3} =$$
   $$= \frac{-16t^2 + 64t\Big|_0^2 + 16t^2 - 64t\Big|_2^3}{3} = \frac{64 + 16}{3} = \frac{80}{3} \text{ feet/second.}$$

   (c) Find an equation for the height $h(t)$ of the ball (remember the ball is thrown from an initial height of 0).

   SOLUTION: We simply need to find the antiderivative of $v(t)$ that has the value 0 at $t = 0$. The antiderivative is $h(t) = -16t^2 + 64t$.

   (d) What is the average height of the ball while it is in the air? First, note that the ball hits the ground when $t = 4$ seconds. So we want to find the average value of $h(t)$ on the interval $[0, 4]$.

   $$\frac{\int_0^4 h(t)\, dt}{4-0} = \frac{\int_0^4 -16t^2 + 64t\, dt}{4} = \frac{\frac{-16}{3}t^3 + 32t^2\Big|_0^4}{4} = \frac{\frac{-1024}{3} + 512}{4} = \frac{-1024 + 1536}{12} = \frac{512}{12} = \frac{128}{3} \text{ feet.}$$

2. (10 points) Estimate the area under the graph of $y = x^3 + 2$ between $x = 1$ and $x = 9$ using four intervals and a left-hand sum.

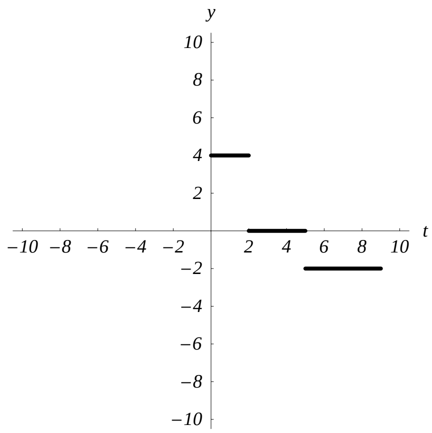
SOLUTION: We need to make four rectangles. They should all have width $\Delta x = \frac{b-a}{n} = \frac{9-1}{4} = 2$. Therefore, the first rectangle will have its base on the interval $[1, 3]$, the second on $[3, 5]$, the third on $[5, 7]$, and the fourth $[7, 9]$.

We will use left endpoints to find the heights of the rectangles. So the first rectangle will have height $f(1)$, the second will have height $f(3)$, the third will have height $f(5)$, and the fourth will have height $f(7)$. So the estimation (using $A = lw$) will be:

$2(f(1)) + 2(f(3)) + 2(f(5)) + 2(f(7)) = 2(f(1) + f(3) + f(5) + f(7)) = 2((1^3 + 2) + (3^3 + 2) + (5^3 + 2) + (7^3 + 2)) = 2(3 + 29 + 127 + 345) = 2(504) = 1108.$

3. (12 points) At noon, Gob rides from his home to the banana stand on his Segway. Since it is downhill, he rides at 4 miles per hour, and makes the trip in two hours. He then spends three hours in front of the banana stand performing magic. Finally, he heads home. His trip home is uphill, so he can only travel at 2 miles per hour, and it takes him 4 hours. Here is a graph of his velocity $v(t)$, where $t = 0$ corresponds to noon:

Graph of Gob's velocity, $v(t)$:



(a) Express the net change in Gob's position from $t = 0$ to $t = 4$ using integral notation (you don't have to find the net change in position, just write down an integral that represents the net change).

SOLUTION: $\int_0^4 v(t)\, dt$.

(b) Find the net change in Gob's position from $t = 0$ to $t = 6$ (your answer should be a number, don't forget the units).

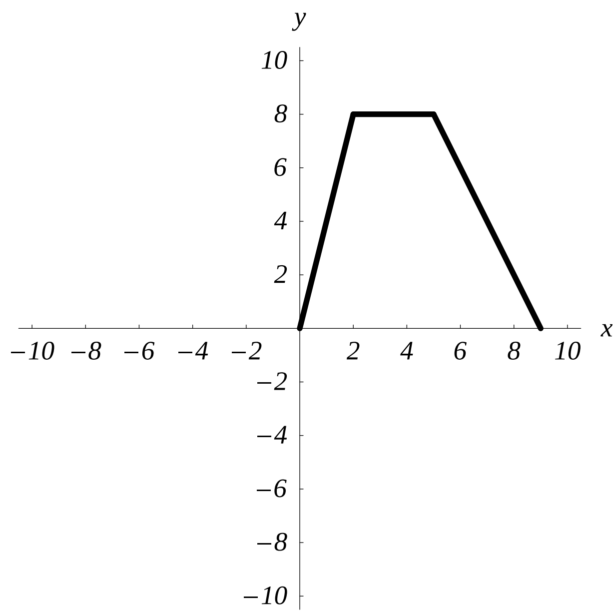
SOLUTION: From $t = 0$ to $t = 2$, we have a rectangle with width 2, height 4, and therefore it has area equal to 8. From $t = 2$ to $t = 5$, there is no new area. From $t = 5$ to $t = 6$, we have a rectangle of area -2. Therefore, the net change during this time is $8 + 0 + (-2) = 6$.

(c) Write a piece-wise defined formula describing Gob's position, $p(x)$, as a function of time x (your formula should *not* contain an integral sign $\int$).

SOLUTION: The formula is that $f(x) = 4x$ if $0 \leq x \leq 2$, $f(x) = 8$ if $2 \leq x \leq 5$, and $f(x) = 8 - 2(x - 5)$ if $5 \leq x \leq 9$.

(d) Graph Gob's position function, $p(t)$, on the axes below. Make your graph as precise as possible.

SOLUTION:



4. (12 points) Below is a list of values for the function $f(x)$. Note that $f(x)$ has a continuous derivative $f'(x)$:

$f(1) = 3$
$f(-1) = -5$
$f(0) = 2$
$f(2) = 7.5$
$f(8) = 0$
$f(-4) = 1$

(a) Find $\int_2^8 f'(x)\,dx$.

SOLUTION: Using the Fundamental Theorem of Calculus Part II, we get that this integral is $\int_2^8 f'(x)\,dx = f(8) - f(2) = 0 - 7.5 = -7.5$

(b) Find $\int_{-1}^{-4} f'(x)\,dx$.

SOLUTION: $\int_{-1}^{-4} f'(x)\,dx = f(-4) - f(-1) = 1 - (-5) = 6.$

(c) Assume that $\int_2^{10} f'(x)\,dx = 7.7$. Find $\int_8^{10} f'(x)\,dx$.

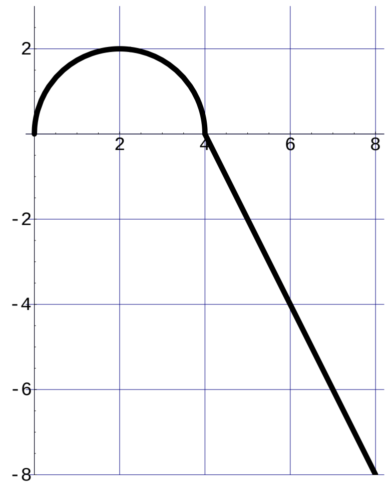
SOLUTION: $\int_8^{10} f'(x)\,dx = \int_2^{10} f'(x)\,dx - \int_2^8 f'(x)\,dx = 7.7 - \int_2^8 f'(x)\,dx = 7.7 - (-7.5) = 15.2$

(d) An antiderivative of $g(x) = \tan x$ is $G(x) = \ln|\sec x|$. Please circle the answer below that best describes $\int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \tan x\,dx$. *Hint:* Draw a picture before you answer this question.

i. $\ln|\sec \frac{3\pi}{4}| - \ln|\sec \frac{\pi}{6}|$
ii. $-(\ln|\sec \frac{3\pi}{4}| - \ln|\sec \frac{\pi}{6}|)$
iii. It cannot be determined from methods we have learned in class.

SOLUTION: Since $\tan x$ is not continuous on this interval (it has a vertical asymptote at $x = \frac{\pi}{2}$), we cannot determine the integral from the methods we have learned in Math Xb.

5. (12 points) The rate that water is entering a tank is given by the function $f(t)$ drawn below. The function $f(t)$ is measured in gallons/minute and t is measured in minutes past noon.



(a) Find $\int_2^6 f(t)dt$. What does this quantity represent in terms of the physical situation?

SOLUTION: This integral represents the net change in the amount of water from 12:02 pm until 12:06 pm.

(b) In terms of the physical situation of the water in the tank, what does the area function ${}_0A_f(x) = \int_0^x f(t)dt$ represent for $x > 0$?

SOLUTION: This function represents the net change in the amount of water from noon until x minutes after noon.

(c) For what values of x is ${}_0A_f(x)$ decreasing?

SOLUTION: This function will decrease when we start accumulating negative area. In other words, it will decrease when $f(t)$ is negative. In other words, it decreases on the interval $[4, 8]$.

(d) Find $\frac{d}{dx}\int_2^x f(t)dt$ at $x = 6$.

SOLUTION: The Fundamental Theorem of Calculus Part I tell us that (since $f(t)$ is continuous) the derivative of $\int_2^x f(t)dt$ is $f(x)$. To find out what this is at $x = 6$, we simply need to find $f(6)$. We use the graph to see that $f(6) = -4$.