Name:____________________________________

Math Xb Midterm 1 Part 1
Spring 2006

Please circle your section:

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Part One Scores

Problem Number	Possible Points	Score
1	6	
2	12	
3	8	
Total	26	

Midterm I Scores

	Possible Points	Score
Part One	26	
Part Two	74	
Total	100	

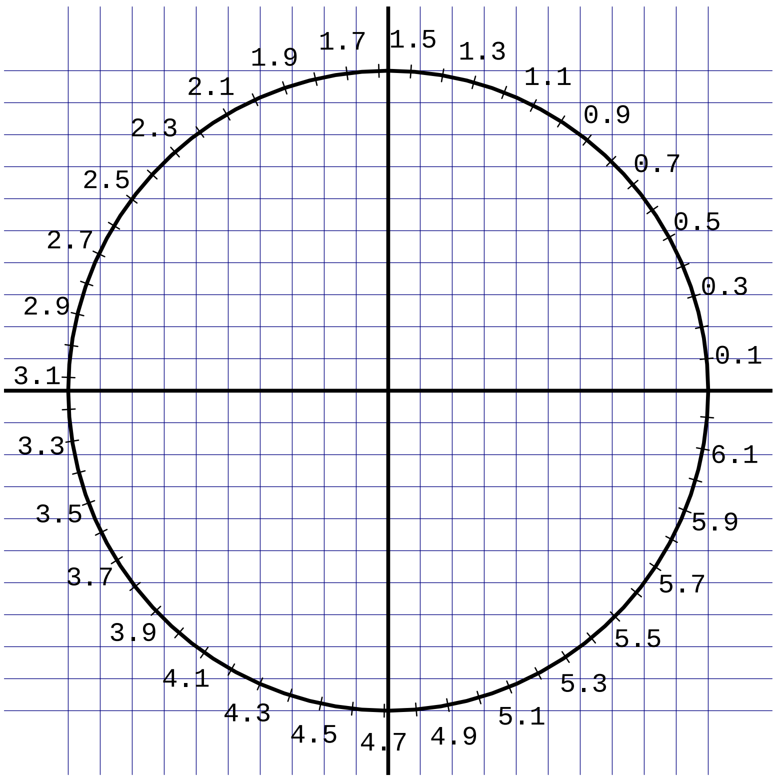
Directions—Please read carefully!

You *are not allowed* to use a calculator or any other aids on this part of the exam. When you are finished with this part of the exam, you may turn it in to the proctor and receive the second part of the exam. Once you have turned in this first part of the exam, you may not look at it again, so be sure you have finished it completely before turning it in.

To receive full credit on a problem, you will need to justify your answers carefully—**unsubstantiated answers will receive little or no credit** (except if the directions for that question specifically say no justification is necessary). Be sure to **write neatly—illegible answers will receive little or no credit**. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers.

Try to finish the first part of the exam within 20 minutes in order to leave time to complete the second part of the exam. The second part of the exam consists of seven problems, and you may use a calculator on the second part of the exam. **Good luck!**

1. (6 points, 1 point each) Use the calibrated unit circle below to evaluate each of the following expressions. You do not need to show your work – no justification of your answers is necessary. Note that each square on the grid below measures 0.1×0.1. Round your answers to the nearest tenth.



(a) $\sin 0.7$

SOLUTION: To find this, we use the fact that $\sin\theta$ is defined to be the y-coordinate of the point on the unit circle that is θ units along the circumference from the point $(1, 0)$. So we trace the circumference of the circle until we reach 0.7. Then we look at what the y-coordinate of the point is. In this case, the y-coordinate is approximately 0.6. So $\sin 0.7 \approx 0.6$.

(b) $\cos 0.7$

SOLUTION: To find this, we use the fact that $\cos\theta$ is defined to be the x-coordinate of the point on the unit circle that is θ units along the circumference from the point $(1, 0)$. So we trace the circumference of the circle until we reach 0.7. Then we look at what the x-coordinate of the point is. In this case, the x-coordinate is approximately 0.8. So $\cos 0.7 \approx 0.8$.

(c) $\sin^{-1} 0.3$

SOLUTION: To find this, we use the fact that $\sin^{-1} y$ is defined to be the distance traveled along the circumference of the unit circle that yields a y coordinate of y. Since this distance is not unique, we refine this definition to only include the distance that lies in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. So we go up to 0.3 on the y-axis, and note that there are two points on the unit circle that have y-coordinate 0.3. The first is to the right, and is the point that is approximately 0.3 units along the circumference of the circle. The second is to the left of the y-axis, and is approximately 2.8 units along the circumference of the unit circle. However, we must choose the value that is between $[-\frac{\pi}{2}, \frac{\pi}{2}] \approx [-1.57, 1.57]$. The former point fits this description, so we conclude that $\sin^{-1} 0.3 \approx 0.3$.

(d) $\cos^{-1} 0.3$

SOLUTION:To find this, we use the fact that $\cos^{-1} x$ is defined to be the distance traveled along the circumference of the unit circle that yields a x coordinate of x. Since this distance is not unique, we refine this definition to only include the distance that lies in the interval $[0, \pi]$. So we go over to 0.3 on the x-axis, and note that there are two points on the unit circle that have x-coordinate 0.3. The first is above the x-axis, and is the point that is approximately 1.3 units along the circumference of the circle. The second is below the x-axis, and is approximately 5.0 units along the circumference of the unit circle. However, we must choose the value that is between $[0, \pi] \approx [0, 3.14]$. The former point fits this description, so we conclude that $\cos^{-1} 0.3 \approx 1.3$.

(e) $\sin^{-1}(-0.3)$

SOLUTION: We will find the answer as described above, and we get two possible points: either 3.4 or 6.0. However, notice that neither of these numbers lies in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}] \approx [-1.57, 1.57]$. However, notice that instead of going counterclockwise around circumference 6.0 units, we could go around the circle clockwise approximately 0.3 units. Since we are going clockwise rather than counterclockwise, we would describe this point as being -0.3 units along the circumference of the unit circle from our starting point of $(1, 0)$. But -0.3 lies in our allowable interval of $[-\frac{\pi}{2}, \frac{\pi}{2}] \approx [-1.57, 1.57]$, so we conclude that $\sin^{-1}(-0.3) \approx -0.3$.

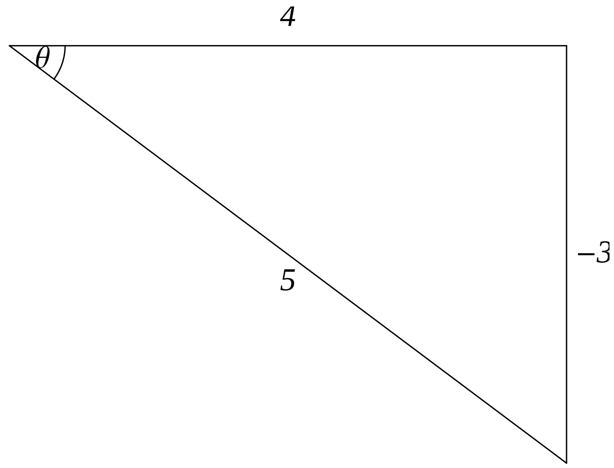
(f) $\cos^{-1}(-0.3)$

SOLUTION: We will find the answer as described above, and we get two possible points: either 1.9 or 4.4. Since 1.9 lies in our allowable interval of $[0, \pi]$, we conclude that $\cos^{-1}(-0.3) \approx 1.9$.

2. (12 points, 3 points each)

(a) Find $\cos\left(\sin^{-1}\left(\frac{-3}{5}\right)\right)$

SOLUTION: We will think of this problem in terms of a right triangle, and use SOH CAH TOA. Since we are considering something involving $\sin^{-1}\left(\frac{-3}{5}\right)$ (which is interpreted as an angle - call it θ - of the right triangle that gives a sine of $\frac{-3}{5}$), we can consider a right triangle with "opposite" side equal to -3 and a hypotenuse of 5. Imagine it in the fourth quadrant of the Cartesian plane:



Then we have reduced the question to: "What is the cosine of θ? Again, using SOH CAH TOA, we get that the answer is $\frac{4}{5}$.

(b) Find $\arctan\left(\frac{-1}{\sqrt{3}}\right)$

SOLUTION: This question is equivalent to: "What angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ has a tangent of $\frac{-1}{\sqrt{3}}$?" This is a common value for tangent to have, so we should know this almost immediately. We know that $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$. Since we can interpret the value of the tangent function as the slope of the line from the origin to a point on the unit circle, we simply need to find a line with the opposite slope as the line from the origin to the point on the unit circle that is $\frac{\pi}{6}$ along the circumference of the unit circle. We can do this if we reflect that line along the x-axis. Then we note that this reflected line hits the circumference at the point $\frac{-\pi}{6}$, which lies in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. So $\arctan\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$.

(c) If $g(x) = \sin\big(\cos(3x^2+12)\big)$, find $g'(x)$.

SOLUTION: This is just applying the chain rule twice:

$g'(x) = \cos\big(\cos(x^2+12)\big) \cdot (\frac{d}{dx}\cos(3x^2+12)) = \cos\big(\cos(x^2+12)\big) \cdot (-\sin(3x^2+12)) \cdot (\frac{d}{dx}3x^2+12) = \cos\big(\cos(x^2+12)\big) \cdot (-\sin(3x^2+12)) \cdot (6x)$.

(d) If $f(x) = x^{x^2+x+1}$, find $f'(x)$.

SOLUTION: Since we have a variable in the exponent of the function, it is useful to take the natural log of both sides of the equation and use the properties of logarithms to simplify:

$$\ln f(x) = \ln(x^{x^2+x+1})$$
$$\ln f(x) = (x^2 + x + 1)\ln x$$

Now we can take the derivative of both sides by using the chain rule on the left side of the equation and the product rule on the right side:

$$\frac{1}{f(x)} f'(x) = (\frac{d}{dx} x^2 + x + 1)\ln x + (x^2 + x + 1)(\frac{d}{dx}\ln x).$$
$$\frac{1}{f(x)} f'(x) = (2x + 1)\ln x + (x^2 + x + 1)\frac{1}{x}.$$

Now we can solve for $f'(x)$ by multiplying both sides of the equation by $f(x)$ and replacing $f(x)$ with x^{x^2+x+1}:

$$f'(x) = f(x)\left((2x + 1)\ln x + (x^2 + x + 1)\frac{1}{x}\right)$$
$$f'(x) = x^{x^2+x+1}\left((2x + 1)\ln x + (x^2 + x + 1)\frac{1}{x}\right)$$

3. (8 points, 4 points each)

(a) Find all solutions to the equation $\sin 2x \cos x = \frac{1}{2} \sin 2x$ for $x \in [0, 2\pi]$. (Your answer should *not* involve any inverse trigonometric functions.)

SOLUTION: When solving an equation, we often want to make one side of the equation equal to zero. This is the case here:

$$\sin 2x \cos x - \frac{1}{2} \sin 2x = 0$$
$$\sin 2x(\cos x - \frac{1}{2}) = 0$$

Since we have two things multiplying together to give us zero, we know that one of them has to be equal to zero already. So either $\sin 2x = 0$ or $\cos x - \frac{1}{2} = 0$.

If $\sin 2x = 0$, then $2x$ could be any of the values $...-3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, ...$ Therefore, x could be any of the values $... -\frac{3}{2}\pi, -\pi, -\frac{1}{2}\pi, 0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi, 2\pi, \frac{5}{2}\pi, 3\pi, ...$ However, the only values here that lie in our interval $[0, 2\pi]$ are $0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi$, and 2π. Therefore, these are the only possible solutions to $\sin 2x = 0$ for this problem.

If $\cos x - \frac{1}{2} = 0$, then $\cos x = \frac{1}{2}$. There are two solutions to this equation on the unit circle - one in the first quadrant, and one in the fourth quadrant. These two solutions can be described as $\frac{\pi}{3} + 2\pi n$ and $\frac{5\pi}{3} + 2\pi n$ for any integer n. However, since we are only allowed values in $[0, 2\pi]$ for this problem, we see that the only solutions that we are allowed right now are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

Combining our two sets of solutions, we get that the solutions to $\sin 2x \cos x = \frac{1}{2} \sin 2x$ for $x \in [0, 2\pi]$ are: $0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi, 2\pi, \frac{\pi}{3}$, and $\frac{5\pi}{3}$.

(b) Find all solutions to the equation $\sin 3x = 0.2$ for x any real number. (Your answer *must* involve the function arcsin.)

SOLUTION: There are two places on the unit circle where $\sin \theta = 0.2$; one is in the first quadrant, and one is in the second quadrant. The first quadrant is contained in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and so the arclength/angle can be described by the arcsine function. In particular, this has value $\arcsin 0.2$. However, we add any multiple of 2π to this, since this causes us to end up on the point on the unit circle. So we would end up at this point in the first quadrant if we travel $\arcsin 0.2 + 2\pi n$ where n can be any integer.

The second quadrant is not contained in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so it cannot be directly described by the arcsine function. However, distance you would need to travel from the point $(-1, 0)$ on the x-axis to this point in the second quadrant is exactly the same as the distance you would need to travel from the point $(1, 0)$ to the point in the first quadrant. This distance is described by $\arcsin 0.2$. So this angle can be described by $\pi - \arcsin 0.2$.

Again, we could add any multiple of 2π to this and end up at the same point, so we will end up at this point in the second quadrant if we travel $\pi - \arcsin 0.2 + 2\pi n$ for any integer n.

But we have $\sin 3x$, rather than $\sin\theta$. Therefore, we have two possible solutions:

$$3x = \arcsin 0.2 + 2\pi n$$
$$3x = \pi - \arcsin 0.2 + 2\pi n$$

where n can be any integer. Now, we merely need to divide by 3 to get all of the solutions to our equation:

$$x = \frac{\arcsin 0.2}{3} + \frac{2}{3}\pi n$$
$$x = \frac{\pi}{3} - \frac{\arcsin 0.2}{3} + \frac{2}{3}\pi n$$