Chapter 17: Implicit Differentiation

Logarithmic Differentiation
General Strategy:

1. Take the natural log of both sides of the equation (remember that ln is not defined for negative arguments)

2. Simplify the expression, move down exponents as per rules of logarithms

3. Take the derivative of both sides remembering to use implicit differentiation

4. Solve for the derivative you are looking for

5. Rewrite the expression explicitly

Logarithmic differentiation has two main uses: differentiating an expression with a variable in the base and the exponent, or for differentiating complicated products or quotients since logs can easily simplify them into sums and differences.

Example 1
Find the derivative of $y = (\tan x)^{3x^2+4x}$ with respect to x. If we only look at some interval say $[0, \frac{\pi}{2}]$ then $\tan x > 0$ so we will proceed with logarithmic differentiation following the steps above:

1. $\ln y = \ln(\tan x)^{4x^2+4x}$

2. $\ln y = (3x^2 + 4x)\ln(\tan x)$

3. $\frac{1}{y}y' = (6x + 4)(\tan x) + (3x^2 + 4x)(\sec^2 x)$

4. $y' = y((6x + 4)\tan x + (3x^2 + 4x)\sec^2 x)$

5. $y' = (\tan x)^{3x^2+4x}((6x + 4)\tan x + (3x^2 + 4x)\sec^2 x)$

Example 2
Find the derivative of $y = \frac{f(x)\cdot g(x)}{h(x)\cdot(k(x))}$ For any functions $f, g, h, k(x)$ assuming that $y > 0$. This assumptions lets us use logarithmic differentiation as above:

1. $\ln y = \ln\left(\frac{f(x)\cdot g(x)}{h(x)\cdot(k(x))}\right)$

2. $\ln y = \ln f(x) + \ln g(x) - \ln h(x) - \ln k(x)$

3. $\frac{1}{y}y' = \frac{1}{f(x)}f' + \frac{1}{g(x)}g' - \frac{1}{h(x)}h' - \frac{1}{k(x)}k'$

4. $y' = y(\frac{1}{f(x)}f' + \frac{1}{g(x)}g' - \frac{1}{h(x)}h' - \frac{1}{k(x)}k')$

5. $y' = (\frac{f(x) \cdot g(x)}{h(x) \cdot k(x)})(\frac{1}{f(x)}f' + \frac{1}{g(x)}g' - \frac{1}{h(x)}h' - \frac{1}{k(x)}k')$

Implicit Differentiation

General Strategy:

1. Take the derivative of both sides of the expression remembering what is a function of which variable and using the chain rule as necessary

2. Algebraically solve for the desired derivative.

Example 3

Find $\frac{dy}{dx}$ from the expression of the shifted circle $(x-a)^2 + (y-b)^2 = c^2$ for some constants a, b and c.

1. $x^2 - 2ax + a^2 + y^2 - 2by + b^2 = c^2$ Expanding the squared quantities

2. $2x - 2a + 2y\frac{dy}{dx} - 2b\frac{dy}{dx} = 0$ Taking the derivative with respect to x

3. $\frac{dy}{dx}(2y - 2b) = 2a - 2x$ Collecting $\frac{dy}{dx}$ terms

4. $\frac{dy}{dx} = \frac{a-x}{y-b}$ Solving for the derivative

Chapter 18: Geometric Sums and Series

Definition
A geometric sum can be written in the form:
$a + ar + ar^2 + ar^3 + \ldots + ar^n$ for some constant ratio r

The sum of the first n terms of a geometric series is: $S = \frac{a - ar^{n+1}}{1-r}$. This formula applies only to geometric series (those that have terms with a constant ratio), using this formula on a non-geometric sum is incorrect.

If we are interested in infinite geometric series, we can take the limit as $n \to \infty$, so the sum of the infinite geometric series converges to $S = \frac{a}{1-r}$ when $|r| < 1$ but diverges when $|r| \geq 0$.

Example 1
What is the sum of the following series: $\frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \ldots$?
First we see that the constant ratio is $r = -\frac{1}{2}$ and that $a = \frac{3}{2}$. We see that this is an infinite geometric series which will converge because $|r| < 1$ so now according to our formula above: $S = \frac{a}{1-r} = \frac{\frac{3}{2}}{1 + \frac{1}{2}} = 1$

Summation Notation
The summation notation is a method for writing sums concisely. It will require that we create an index to count from the first term to the last. Usually this index will also be involved in specifying the terms which we are adding. Consider the following non-geometric sum: $2^3 + 3^4 + 4^5 + \ldots + 100^{101}$. We see that the base is increasing by one each term, and so is the exponent. So $\sum_{i=2}^{100} i^{i+1} = 2^3 + 3^4 + 4^5 + \ldots + 100^{101}$ After writing a particular sum in summation notation, it is a good idea to make sure the first and last terms match up. So our sum starts at $i = 2$ which gives 2^{2+1} as a first term and ends at $i = 100$ which gives 100^{100+1} as a last term.

Pay particular attention to the applications of geometric sums in chapter 18.5 as they can be tricky.