Name: SOLUTIONS

Math Xb Midterm Examination II
Thursday, April 15, 2004

Please circle your section:

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Problem Number	Possible Points	Score
1	10	
2	10	
3	10	
4	12	
5	10	
6	12	
7	14	
8	12	
9	10	
Total	100	

Directions—Please Read Carefully! You have two hours to take this midterm. Make sure to use correct mathematical notation. Any answer in decimal form must be accurate to two decimal places, unless otherwise specified. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You do not have to answer the problems in any particular order, so move to another problem if you find you are stuck or spending too much time on a single problem. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit (except if the directions for that question specifically say no justification is necessary). Please be sure to write neatly—illegible answers will receive little or no credit. If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. You may use a calculator on this exam, but no other aids are allowed. **Good Luck!**

1. Let $f(x) = \sin x + \cos x$.

(a) At what exact value(s) of x on the interval $[2\pi, 3\pi]$ does f have an absolute maximum?

$$f'(x) = \cos x - \sin x = 0$$

$$\frac{\sin x}{\cos x} = \frac{\cos x}{\cos x}$$

$$\tan x = 1 \quad \text{and } x \in [2\pi, 3\pi]$$

$$\Rightarrow \quad x = 2\pi + \frac{\pi}{4} = \frac{9\pi}{4}$$

critical pts : $2\pi, \frac{9\pi}{4}, 3\pi$

$$f(2\pi) = \sin 2\pi + \cos 2\pi = 0 + 1 = 1$$

$$f\left(\frac{9\pi}{4}\right) = \sin\frac{9\pi}{4} + \cos\frac{9\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

$$f(3\pi) = \sin 3\pi + \cos 3\pi = 0 - 1 = -1$$

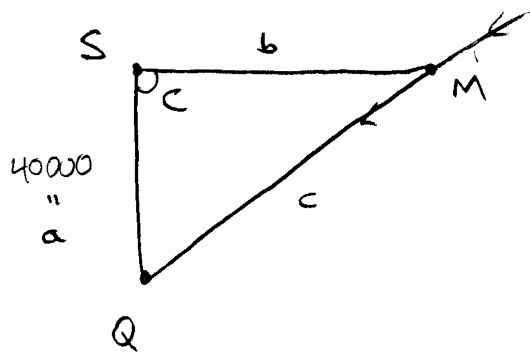
$\boxed{\text{abs. max at } x = 9\pi/4}$

(b) At what exact value(s) of x on the interval $[2\pi, 3\pi]$ does f have an absolute minimum?

abs. min at
$x = -1$

2. A NASA satellite observes a small meteoroid heading directly for the city of Quito, Ecuador. At the time of initial observation, the distance between the meteoroid and the satellite is 30,000 kilometers and ~~increasing~~ decreasing at a rate of 10 kilometers per second. Using the satellite as the vertex, the angle between the meteoroid and Quito is $\frac{\pi}{2}$ radians and decreasing at a rate of 0.01 radians per second. How fast is the distance between the meteoroid and Quito changing? Assume that the satellite maintains a position of 40,000 kilometers directly above Quito.

(Recall that the Law of Cosines states that $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the measure of the angle between two sides of length a and b, and c is the length of the side opposite that angle.)



$b = 30000$

$\frac{db}{dt} = -10$ $\qquad\qquad$ $\frac{dc}{dt} = ?$

$C = \pi/2$

$\frac{dC}{dt} = -.01$

$c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = (40000)^2 + b^2 - 2(40000)b\cos C$

$2c\frac{dc}{dt} = 2b\frac{db}{dt} - 80000\left(-b\sin C\frac{dC}{dt} + \frac{db}{dt}\cos C\right)$

$2(50000)\frac{dc}{dt} = 2(30000)(-10) - 80000\Big(-30000\underbrace{\sin \pi/2}_{=1}(-.01)$
$+ (-10)\underbrace{\cos \pi/2}_{=0}\Big)$

$\frac{dc}{dt} = \frac{-600000 - 24000000}{100000}$

$= \boxed{-246 \text{ km/s}}$

$c^2 = (40000)^2 + b^2 - 80000b\cos C$

$= (40000)^2 + (30000)^2 - 80000(30000)\underbrace{\cos\frac{\pi}{2}}_{=0}$

$= (40000)^2 + (30000)^2$

$= 1600000000 + 900000000$

$= 2500000000$

$c = 50000$

3. Suppose that $\cot y = x$. Use implicit differentiation to show that

$$\frac{dy}{dx} = -\frac{1}{1+x^2}.$$

Show all steps clearly to receive full credit.

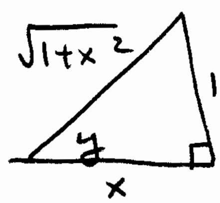
$$\cot y = x$$

$$-\csc^2 y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\csc^2 y} = -\sin^2 y$$

$$\cot y = x$$

$$\Rightarrow \tan y = 1/x$$



$$\Rightarrow \sin y = 1/\sqrt{1+x^2}$$

$$\Rightarrow \frac{dy}{dx} = -(\sin y)^2 = -\left(\frac{1}{\sqrt{1+x^2}}\right)^2$$

$$= \boxed{-\frac{1}{1+x^2}}$$

4. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

is said to be of indeterminate form $\frac{\infty}{\infty}$.

(a) Give an example of particular functions $f(x)$ and $g(x)$ such that $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is of indeterminate form $\frac{\infty}{\infty}$ *and* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

$f(x) = x$

$g(x) = x^2$

(b) Give an example of particular functions $f(x)$ and $g(x)$ such that $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is of indeterminate form $\frac{\infty}{\infty}$ *and* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 12$.

$f(x) = 12x$

$g(x) = x$

(c) Give an example of particular functions $f(x)$ and $g(x)$ such that $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is of indeterminate form $\frac{\infty}{\infty}$ *and* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$.

$f(x) = x^2$

$g(x) = x$

5. Determine the exact value of the following limit.

$$L = \lim_{x\to\infty}\left(1+\frac{4}{x}\right)^{7x} = \text{" } 1^{\infty} \text{ "}$$

$$\ln L = \ln\left(\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^{7x}\right)$$

$$= \lim_{x\to\infty} \ln\left(1+\frac{4}{x}\right)^{7x}$$

$$= \lim_{x\to\infty} 7x\cdot\ln\left(1+\frac{4}{x}\right) = \text{" } \infty\cdot 0 \text{ "}$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{4}{x}\right)}{\frac{1}{7x}} = \text{" } \frac{0}{0} \text{ "}$$

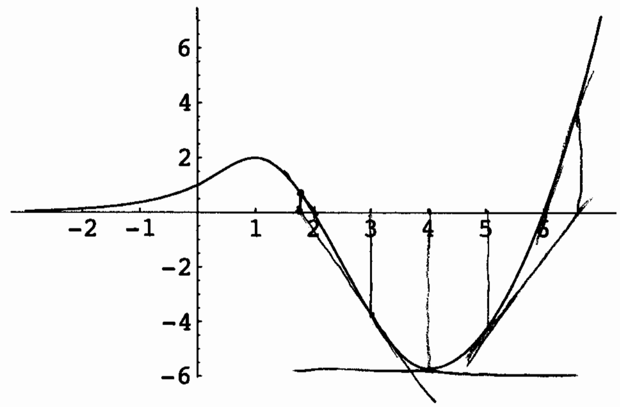
$$\overset{L}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{4}{x}}\cdot\frac{-4}{x^2}}{-\frac{1}{7x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{-4}{1+\frac{4}{x}}}{-1/7} = \lim_{x\to\infty} \frac{28}{1+4/x} = 28$$

$$\Rightarrow \ln L = 28$$

$$\boxed{L = e^{28}}$$

6. Suppose you are trying to use Newton's Method to determine the roots of the function whose graph is given below.



For each of the following initial guesses, determine whether or not Newton's Method will converge to a root. If it will converge, identify the root to which it converges. If it will not converge, explain why not.

(a) $x_0 = 3$

converges to $x = 2$

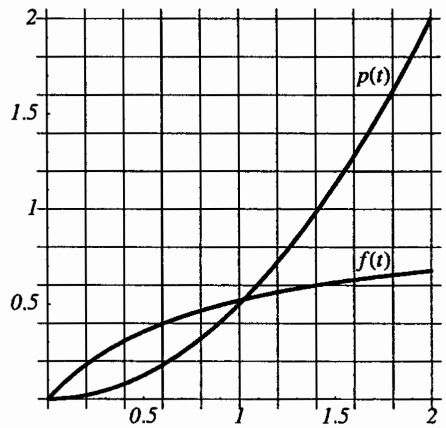
(b) $x_0 = 4$

does not converge: The tangent line at $x = 4$ is horizontal so it never intersects the x-axis and so there is no guess x_1.

(c) $x_0 = 5$

converges to $x = 6$

7. Let $p(t)$ be the rate of growth (in meters per year) of a particular pine tree. Let $f(t)$ be the rate of growth (in meters per year) of a particular fir tree. The graphs of $p(t)$ and $f(t)$ are given below. Assume that the two trees were of the same height at $t = 0$.



(a) Express the height of the pine tree after six months as a definite integral.

$\int_0^{1/2} p(t)\,dt$

(b) Which tree is taller after 1 year?

fir because $\int_0^1 f(t)\,dt > \int_0^1 p(t)\,dt$

(c) Which tree is taller after 2 years?

pine because $\int_0^2 p(t)\,dt > \int_0^2 f(t)\,dt$

(d) At approximately what time are the trees growing at the same rate? Justify your answer.

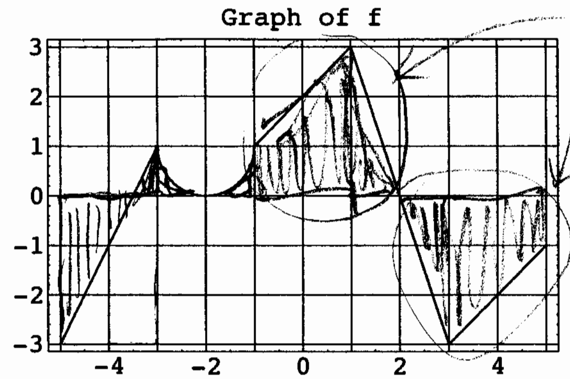
after about 1 year ; since $f(t)$ and $p(t)$ give the rates of growth and ~~they~~ intersect around $t = 1$ (also at $t=0$)

(e) At approximately what time are the trees the same height? Justify your answer.

when the area under $p(t)$ equals the area under $f(t)$ $\Rightarrow$ around 1.5 or 1.6 years

(also at $t = 0$)

8. The graph of f, given below, is made up of straight lines and a semicircle.



correct

(a) Estimate $\int_{-5}^{5} f(x)\,dx$ using a left-hand sum with 5 subintervals.

$$= f(-5)\cdot 2 + f(-3)\cdot 2 + f(-1)\cdot 2 + f(1)\cdot 2 + f(3)\cdot 2$$

$$= (-3)\cdot 2 + 1\cdot 2 + 1\cdot 2 + 3\cdot 2 + (-3)\cdot 2$$

$$= -6 + 2 + 2 + 6 - 6 = \boxed{-2}$$

(b) Estimate $\int_{-5}^{5} f(x)\,dx$ using a right-hand sum with 5 subintervals.

$$= f(-3)\cdot 2 + f(-1)\cdot 2 + f(1)\cdot 2 + f(3)\cdot 2 + f(5)\cdot 2$$

$$= 1\cdot 2 + 1\cdot 2 + 3\cdot 2 + (-3)\cdot 2 + (-1)\cdot 2$$

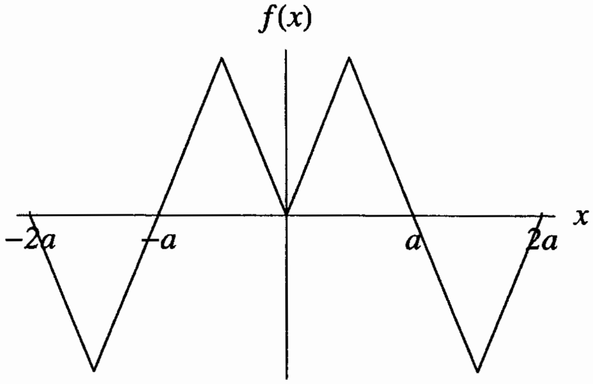
$$= 2 + 2 + 6 - 6 - 2 = \boxed{2}$$

(c) Find the exact value of $\int_{-5}^{5} f(x)\,dx$.

$$-\tfrac{1}{2}(3)(3/2) + \tfrac{1}{2}(\tfrac{1}{2})(1) + (2)(1) - \tfrac{1}{2}\pi(1)^2$$

$$= -\frac{9}{4} + \frac{1}{4} + 2 - \frac{\pi}{2} = \boxed{\frac{-\pi}{2}}$$

9. The graph of a function f is given below.



Use the graph to put the following integrals in ascending order, from most negative to most positive. Express your answer as something like "$a < b < c < d < e < f$."

(a) $\int_0^a f(x)\,dx$ = 1 triangle

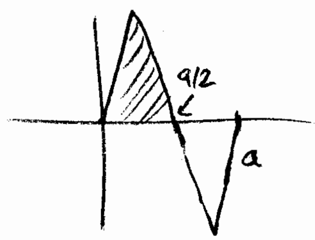
(b) $\int_0^{2a} f(x)\,dx$ = 0 triangles

(c) $\int_a^{2a} f(x)\,dx$ = −1 triangle

(d) $\int_{-a}^{a} -f(x)\,dx$ = −2 triangles

(e) $\int_0^{a/2} f(2x)\,dx$ = $\frac{1}{2}$ triangle

(f) $\int_{-2a}^{2a} |f(x)|\,dx$ = 4 triangles



f(2x)

$\boxed{d < c < b < e < a < f}$