Equations in algebra and topology
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for the Birkhoff Memorial Symposium
Cambridge, Massachusetts
April 1, 1997

This is a loose outline of general algebra and its history, with emphasis on the founding contributions of Garrett Birkhoff. I finish with some remarks on my own recent work. The hour talk will cover a subset of this material. 

Abstract of the talk: In the early 1930's, Garrett Birkhoff introduced the notion a variety $\mathcal{V}$, i.e. the class of all algebras (in the general sense) that model a fixed set $\Sigma$ of equations, and proved his famous theorem characterizing such equational classes being classes closed under the formation of subalgebras, homomorphic images and products. Ever since then (and more intensely since 1970), such classes have been actively studied — both in general and in many particular examples.

In this talk we will survey some of this work, mentioning important results and problems of Tarski, Jonsson, McKenzie, Baker. We will also include some recent material on the modeling of the equations $\Sigma$ by continuous operations on a topological space $A$. (Birkhoff was interested in this question as well.) Some recent results of the author say that many simple spaces $A$, e.g. a 2-sphere or surface of genus 2, cannot continuously model any except the most trivial of equations $\Sigma$.

## 0.1 Garrett Birkhoff and universal algebra

Birkhoff's *Lattice Theory* [5] not only presented his vision of that subject; it also defined his relation to general (or universal) algebra, and gave the first coherent account of that subject. There were three editions[1]: 1940, 1948 and 1967. The revisions were so thorough that it is hard to correlate parts of one edition with parts of another; each of the two earlier editions contains some material not found in the final (1967) edition. I suppose it can be regarded as a compliment to the book, that when I was in residence at Harvard, both the first and second editions were routinely reported missing (stolen?) from the Harvard Library System. (The third was not yet in print.)

Later books were written on general algebra by G. Grätzer [18], R. S. Pierce [39], P, M. Cohn [11], S. N. Burris and H. P. Sankappanavar [9], B. Jónsson [22], and R. N. McKenzie, G. F. McNulty and W. Taylor [15].

## 0.2 Free algebras

Prior to Birkhoff's work in the thirties, free groups were known, and free groups in a variety (i.e. free relative to some equational constraint) were starting to be known. Birkhoff and others (notably Whitman) worked on free lattices. Birkhoff was one of the first to give a coherent general definition of free algebras for a class, and to give a general proof of their existence. (The proof of Theorem 1 was later given much more generally as the 'Adjoint functor theorem' of category theory—see, e.g., page 84 of [16].)

Let $\mathcal{K}$ be a class of algebras of a fixed type $\rho$, and let $X$ be any set. A **$\mathcal{K}$-free algebra on** $X$ (denoted $\mathbf{F}_{\mathcal{K}}(X)$) is defined to be an algebra $\mathbf{A} \in \mathcal{K}$ with $X \subseteq A$ which satisfies the following universal mapping property:

> If $\mathbf{B} \in \mathcal{K}$, and $f_0 : X \longrightarrow B$ is any function from $X$ to the universe of $\mathbf{B}$, then there exists a unique homomorphism $f : \mathbf{A} \longrightarrow \mathbf{B}$ such that $f|_X = f_0$.

**Theorem 1 (G. Birkhoff)** *If $\mathcal{K}$ is any non-trivial class of algebras (all of the same similarity type) closed under the formation of subalgebras and products, then $\mathbf{F}_{\mathcal{K}}(X)$ exists for every $X$.*

## 0.3 Varieties

An **equationally defined class**, alias a **variety**, is a class $\mathcal{V}$ of algebras (all of the same type $\rho$) for which there exists a set $\Sigma$ of equations with

$$\mathcal{V} \;=\; \mathrm{Mod}\,\Sigma \;=\; \big\{\mathbf{A} : (\forall e \in \Sigma)\, \mathbf{A} \models e\big\}.$$

The Rosetta Stone of Birkhoff's work in general algebra was the following theorem, which says that a mere closure condition implies the existence of certain equations.

[1] Page references in this document are to the third edition.

**Theorem 2 (G. Birkhoff, [2])** *$\mathcal{V}$ is a variety iff $\mathcal{V}$ is a class of algebras (of the same type) that is closed under the formation of products, subalgebras, and homomorphic images.*

(The early paper [2] also appears in Birkhoff's selected works [7].) Obviously Theorem 2 also appears in various books on general algebra, such as [9], [11], [18], [22], [32], [39]. The overall style of this theorem was unknown when the theorem was first published. There came to be many similar theorems (all very interesting in their own right) in the allied field of *model theory*—see for instance [10]. Concerning varieties themselves, there are too many papers to cite. Variety theory is sometimes called *Equational Logic* [40, 43, 34]

Birkhoff was interested in *subvariety lattices*. W. A. Lampe proved [25] that lattices in a certain large collection cannot be isomorphic to the lattice of subvarieties of any variety. For instance, the five-element modular lattice $\mathbf{M}_3$ is of this type. There is no variety $\mathcal{V}$ whose subvarieties form a lattice isomorphic to $\mathbf{M}_3$. The proof is clever but elementary; the reader is referred to [25].

The notion of variety is enormously powerful. We refer the reader, for one example only, to the work of Tarski and Givant [41], where it is shown how to express a full set theory as a variety in a certain language.

## 0.4 Word problems and decidability

As soon as you have the notion of a free algebra in a variety, you can ask questions about down-to-earth ways of expressing the free algebra. The usual way is with *words* (formal expressions in the generators).

As Birkhoff emphasized, every word in group theory can be reduced uniquely to a word of the form $x_1^{\pm n_1} x_2^{\pm n_2} \cdots x_k^{\pm n_k}$. This reduction implies a decision procedure: given an equation $\sigma \approx \tau$, reduce both $\sigma$ and $\tau$ and see whether they come out exactly the same. If they do, then $\sigma \approx \tau$ lies in the equational theory of groups, otherwise not. The same sort of procedure works for various other theories, such as semigroups and Boolean algebras (where one can use the famous *disjunctive normal form*). Notice that, conversely, our concrete description of free groups relies on having some such decision procedure. If $\Sigma$ is an undecidable equational theory, then there is really no way to provide an effective description of the free algebras in $\operatorname{Mod}\Sigma$.

Whitman proved that a similar algorithm exists for lattice theory. R. Freese proved in 1980 [13] that the variety of modular lattices does not have a decidable equational theory. This answered Problem 42 on page 157 of *Lattice Theory*.

## 0.5 Crypto-isomorphism (equivalence)

Birkhoff invented and championed a notion of *crypto-isomorphism*, between algebras of different similarity types. The archetypical example is the well-known translation between Boolean algebras and Boolean rings. (In Birkhoff's vision, these two would also be crypto-isomorphic to a Boolean lattice, where complements must be inferred from context (see page 155 of *Lattice Theory*) — but a coherent exposition of this level of generality remains open.) Among modern-day practitioners, crypto-isomorphism has effectively been superseded by the Tarski-Mal'tsev notion of equivalence (and their related notion of an interpretation of one theory in another). See e.g. Tarski [40] for equivalence, or García and Taylor [17] for interpretations.

## 0.6 The lattice of congruence relations; modularity

The great link for Birkhoff — and still today — between general algebra and lattice theory, was the lattice of congruence relations on an algebra. (If $f : \mathbf{A} \longrightarrow \mathbf{B}$ is a homomorphism, then $\{\langle a, a' \rangle : f(a) = f(a')\}$ is an equivalence relation; such equivalence relations are called congruence relations.) For groups, the lattice of congruence relations is isomorphic to the lattice of normal subgroups. Sometimes Birkhoff used the term "structure lattice" in place of congruence lattice.

A great triumph for Birkhoff was to see the lattice of congruence relations as a generalization of Dedekind's lattice (or *Dualgruppe*). The modular law arose in this context (Dedekind to referred to *Dualgruppen von Modultypus*), and continued to play a very strong rôle. The rôle of the modular law was not fully elucidated until H.-P. Gumm, Christian Herrman, et al., developed a new calculation called the *modular commutator*; the subject is very well covered in R. Freese and R. McKenzie [14]. In this context we have all sorts of things like nilpotence, and so on, coming into universal algebra.

In Problem 63 on page 179, Birkhoff asked, "Is the structure lattice of every group monomorphic to the structure of some Abelian group?" A recent paper by ..... affords an answer.

## 0.7 Axiomatizations; group theory

As we already saw, Birkhoff's interests bridged from the abstract to some very known worlds, such as that of group theory. For example, in studying axiom systems for groups, he branched off into axiom systems for such things as quasi-groups and equasigroups (see p. 160 of *Lattice Theory*). (For an up-to-date account of connections and parallels between general algebra and group theory, see McKenzie [28].)

In Problem 51 (page 158), Birkhoff somewhat naïvely asked for all varieties of groups. I know he was pleased at the appearance of Hanna Neumann's book [36] on the subject in 1967 (the same year of publication as the third edition of *Lattice Theory*). (Similarly, Problem 52 asked for all varieties of lattices.)

In 1952 [20] G. Higman and B. H. Neumann had proved that the operation

$x/y = x \cdot y^{-1}$ of group theory is axiomatized by the single equation

$$x\,/\{[((x/x)/y)/z]\,/[((x/x)/x)/z]\} \;\approx\; y.$$

Also Oates and Powell had proved in 1964 [37] that for each finite group $G$ there is a finite set $\Sigma$ of equations such that all the equations true in $G$ are deducible from $\Sigma$. Such a $\Sigma$ is called a finite basis for these equations. Later K. A. Baker (a Ph.D. student of Birkhoff's) showed [1] that a finite algebra in a congruence-distributive variety (see §0.8) also has a finitely based equational theory.

Nevertheless, there exist finite algebras that do not have any finite basis for their equations. The first such algebra was discovered by Roger Lyndon in 1954, and had seven elements. We will describe a three-element algebra (which is the minimal size), given by V. L. Murskiĭ [35] in 1965. It has a single binary operation, given by the following multiplication table:

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 |
| 2 | 0 | 2 | 2 |

McKenzie recently solved a very old and very hard problem of Tarski, proving [29] that there is no algorithm to determine whether a finite algebra has a finite basis for its equations.

## 0.8 Subdirect representation; distributivity

A **subdirect product** of algebras $\mathbf{C}_i$ $(i \in I)$ is an algebra $\mathbf{B} \subseteq \prod_{i\in I} \mathbf{C}_i$ such that $\mathbf{B}$ projects onto $\mathbf{C}_i$ for each $i$. $\mathbf{A}$ is *subdirectly irreducible* iff for any isomorphism $\phi$ of $\mathbf{A}$ with a subdirect product $\mathbf{B}$ of algebras $\mathbf{C}_i$, there is at least one projection $p_i$ of the product onto $\mathbf{C}_i$ such that $p_i \circ \phi$ is an isomorphism of $\mathbf{A}$ with $\mathbf{C}_i$.

**Theorem 3 (G. Birkhoff [4])** *Every algebra $\mathbf{A}$ is isomorphic to a subdirect product of subdirectly irreducible algebras.*

(The early paper [4] also appears in Birkhoff's selected works [7].)

It is not hard to see that if an algebra $\mathbf{A}$ lies in a variety $\mathcal{V}$, then all subdirect factors of $\mathbf{A}$ also lie in $\mathcal{V}$. Therefore, the utility of subdirect representation for a variety $\mathcal{V}$ depends on the class $\mathcal{S}_{\mathcal{V}}$ of subdirectly irreducible algebras in $\mathcal{V}$. The only subdirectly irreducible semilattice, distributive lattice or Boolean algebra is the one with two elements, and so these three varieties have spectacular subdirect representation theorems. (Cf. Stone's Theorem for Boolean algebras or distributive lattices — these results motivated Theorem 3.) On the other hand, the subdirectly irreducible groups (even the finite ones) form an intractable class, which contains the simple groups, and many other groups besides.

A variety $\mathcal{V}$ is called **residually small** iff there is a cardinal number $\kappa$ that bounds the cardinality of subdirectly irreducible algebras in $\mathcal{V}$. Although no precise theorem is available, it seems that varieties that are residually small generally have nicer structure than varieties that are not. Semilattices, distributive lattices and Boolean algebras are residually small—with $\kappa = 2$. On the other hand, neither groups nor lattices form a residually small variety. It is known that *if $\mathcal{V}$ is a residually small variety of type $\rho$, then $\mathcal{V}$ is residually small for $\kappa = 2^{\aleph_0+|\rho|}$* (see [42]).

For any class $\mathcal{K}$ of algebras, the smallest variety $\mathcal{V}$ containing $\mathcal{K}$ is $\mathsf{HSP}\mathcal{K}$, the class of homomorphic images of subalgebras of products families of algebras from $\mathcal{K}$. In 1967 B. Jónsson proved a Lemma that describes all the subdirectly irreducible algebras in $\mathcal{V} = \mathsf{HSP}\mathcal{K}$ , provided that *all congruence lattices of algebras in $\mathcal{V}$ are distributive.* The conclusion of Jónsson's Lemma is that *every subdirectly irreducible algebra of $\mathcal{V}$ lies in $\mathsf{HSU}\mathcal{K}$*, where $\mathsf{U}$ refers to the formation of **ultraproducts**. If $\mathcal{K}$ is a finite set of finite algebras, then the ultraproduct operation is trivial, and so *every subdirectly irreducible algebra of $\mathsf{HSP}\mathcal{K}$ lies in $\mathsf{HS}\mathcal{K}$*. (This finite version was also discovered by A. Pixley.) Thus, in the finite case, a subdirectly irreducible algebra cannot have cardinality greater than the largest member of $\mathcal{K}$, and hence of course we have residual smallness. For an up-to-date survey on the general area of congruence-distributive varieties, see Jónsson [23].

## 0.9 Permutability of congruences; direct factorization

Birkhoff was one of the first to see the importance of having two congruences $\theta$ and $\phi$ *permute* (see pages 161–163 of *Lattice Theory*). This means that $\theta\circ\phi = \phi\circ\theta$, where $\circ$ denotes a relational product (defined like composition of functions). A variety $\mathcal{V}$ is said to have permutable congruences if $\theta\circ\phi = \phi\circ\theta$ for all algebras $\mathbf{A} \in \mathcal{V}$ and for all congruences $\theta$ and $\phi$ on $\mathbf{A}$. Permutability is important because it is an algebraic condition implying modularity, and because it occurs in many natural examples like groups, or even equasigroups.

Permutability also was significant for various results on uniqueness of direct factorizations of algebras. (These results generalized those of Remak, Wedderburn, Krull and Schmidt in the early part of this century.) Birkhoff's general result spliced congruence-permutability into the famous result of Ore in pure lattice theory (see pages 168–169 of *Lattice Theory*). Parallel developments had recently come out of the school of Tarski, most notably the work of Jónsson and Tarski [24]. For an up-to-date statement of results and problems in this general area see McKenzie, McNulty and Taylor [32, Chapter 5].

Birkhoff liked very much the Theorem of Mal'tsev (1954) (see page 163 of *Lattice Theory*), which said that $\mathcal{V}$ is congruence-permutable iff there is a term $p(x, y, z)$ (i.e. $p$ is a polynomial in the operations that define $\mathcal{V}$) such that $\mathcal{V}$

satisfies the equations

$$p(x, x, z) \approx z; \qquad\qquad p(x, z, z) \approx x.$$

Such a term $p$ is called a Mal'tsev term for $\mathcal{V}$. The obvious example is $p(x, y, z) = xy^{-1}z$ in group theory.

Not all Mal'tsev terms are obvious. In a modern tour de force, S. Tschantz proved [51] that permutability holds for the variety defined by these two equations:

$$(xy)(yx) \approx y; \qquad\qquad xx \approx x.$$

Tschantz discovered and proved this result by a computer search, finally discovering that

$$p(x, y, z) \; = \; [((y(yx))((yx)z))((yz)x)][(x(zy))((z(xy))((xy)y))].$$

will serve as a Mal'tsev term.

See Burris [8] for a recent description of the difficulties faced by a computer search for a Mal'tsev term. In fact, G. McNulty proved [33] that there is no algorithm to decide, given an axiomatization of $\mathcal{V}$, whether $\mathcal{V}$ has a Mal'tsev term.

## 0.10 Connections with topology

At various times, and in various directions, Birkhoff sought a theory of topology that would fit the algebraic mold. Even in his seminal paper [2] on general algebra, he could not resist referring to a *Konvergenzraum* (in the sense of Kneser) as an (infinitary, partial) algebra. In 1937, Birkhoff published a paper [3] on the more general Moore-Smith convergence, especially its algebraic aspects. Much later, his student G. A. Edgar showed [12] that (in the correct context), topological spaces themselves form a variety, or equationally defined class. See also N. Weaver [52] for the description of metric spaces as a quasi-variety, or class defined by equational implications.

In [3] (as in the earlier [2]), Birkhoff emphasized the importance of *compatibility* between topology and algebraic structure (continuity of the operations), and began the search for naturally arising occurrences of this compatibility. In other words his main focus there, and later in *Lattice Theory* (see pages 241–251), was to find topologies that are not only compatible with the algebraic structure, but also inherent or *intrinsic* to the algebra.

The prototype for intrinsic topologies was of course the real-number line with its order structure, where basic open sets are defined as intervals. (Frink had a generalization to something called the *interval topology.*) Moreover, in the *complete lattice* $\langle \boldsymbol{R}, \wedge, \vee \rangle$, one could recover the topology by defining $\liminf$ and $\limsup$. (In general, the topology or would-be topology induced through $\liminf$ and $\limsup$ is called the *order topology.*) Ultimately, it has to be said that this program — of finding general constructions of intrinsic topologies — was not very successful.

In a 1973 survey [6, p. 773], Birkhoff wrote

> Thus to young men in 1930, like myself, van der Waerden's book made *classical analysis* ... suddenly seem old and tired. ... The idea that all mathematics could be viewed as topological algebra gained a strong impetus ... .

Although I do not recall his expressly stating that "all mathematics is topological algebra," nevertheless I am sure that he conveyed to me some sense of the passage that I quoted above, and that my love of topological algebra goes back to my contact with Garrett Birkhoff. (My thesis was on the algebraic consequences of having a compatible Hausdorff topology that is compact.) I therefore would like to share with you a recent result of mine in topological algebra. I was looking for this result ever since my time at Harvard. I first formally stated it in the early eighties, and finally solved it in 1996. Once you really see what is wanted, it is not deep.

It has long been known, for many sets $\Sigma$ of equations, that not all topological spaces $A$ can model $\Sigma$ with continuous operations. For example if $A$ is a sphere of even dimension, or an orientable surface of genus 2, then the axioms of group theory (or even H-spaces) cannot be modeled continuously on $A$. Or, as we say, $A$ is *not compatible with* $\Sigma$. In [44] and [48], I proved that the above-mentioned spaces $A$ are not compatible with any equations $\Sigma$ that imply the existence of a Mal'tsev term $p(x, y, z)$ (see §0.9). (As we already mentioned, group theory is one such $\Sigma$.)

For many years I conjectured the stronger result that the above-mentioned spaces are incompatible with any $\Sigma$ that is not trivial in an appropriate sense. Here by a trivial set of equations $\Sigma$, we mean one that can be satisfied by interpreting each function symbol $F_t(x_1, \cdots, x_n)$ by a function $\overline{F_t} : A^n \longrightarrow A$ which is either a constant map or a projection function. (Such interpretations are called *automatic*.) Since there are so many meanings that can be ascribed to "trivial" for sets $\Sigma$ of equations, we instead call such a $\Sigma$ *easily satisfied* or *undemanding*. For example, semigroup theory, defined by the single equational axiom

$$(xy)z \;=\; x(yz) \tag{1}$$

is obviously undemanding — just define the product $x_1x_2$ to be $x_1$ (this is projection to the first coordinate), and you will make equation (1) true (even on a sphere of even dimension).

Thus the problem was either to find an undemanding $\Sigma$ that can be modeled on, say, a sphere of even dimension, or to prove that this is impossible. The problem was was first formally stated (for a number of spaces) on page 38 of [49]. I finally saw the answer one year ago, in the spring of 1996. Here are two of the findings:

**Theorem 4** *(W. Taylor) If $A$ is a path-connected topological space whose cohomology ring (over some integral domain) is generated by an element $a$ of even degree, with $a^2 = 0$, and if $A$ is compatible with $\Sigma$, then $\Sigma$ is undemanding.*

**Theorem 5** *(W. Taylor) If $A$ is the orientable surface of genus 2, and if $A$ is compatible with $\Sigma$, then $\Sigma$ is undemanding.*

**Proof of Theorem 4.** Let $A$ be a space satisfying the conditions of Theorem 4, for example, the 2-sphere $S^2$. We are given continuous operations $\overline{F_t}\colon A^{n(t)} \longrightarrow A$ such that

$$(A, \overline{F_t})_{t\in T} \models \Sigma$$

(i.e., the equations $\Sigma$ are satisfied in $(A, \overline{F_t})_{t\in T}$), and we need to construct automatic interpretations $F'_t$ modeling $\Sigma$. (This will be done on page 11.)

Before we go on let us analyze for a moment what it means for $(A, \overline{F_t})_{t\in T}$ to satisfy $\Sigma$. Every term $\tau$ defines a function $\overline{\tau}\colon A^\omega \longrightarrow A$ in the obvious way. (E.g. if $\tau = F_2(x_0, F_1(x_2, x_3))$, then $\overline{\tau}(a_0, a_1, \cdots) = \overline{F_2}(a_0, \overline{F_1}(a_2, a_3))$. The official definition is obviously recursive.) Very occasionally (e.g. in Lemma 6) we use $\overline{\tau}^N$, which has domain $A^N$ for a finite integer $N$. Now we can give a useful statement of what it means for $(A, \overline{F_t})_{t\in T}$ to satisfy $\Sigma$: for any equation $\sigma \approx \tau$ in $\Sigma$, we have $\overline{\sigma} = \overline{\tau}$ (they are the same function). We come back to this in the last paragraph of the proof (page 11).

**Definition of the co-operations $F_t^\star$.** We begin by letting $H$ stand for $H^\star(A, R)$, and then defining co-operations

$$H \xrightarrow{F_t^\star} \bigotimes^{n(t)} H$$

via

$$F_t^\star \;=\; \times_{n(t)}^{-1} \circ H^\star(\overline{F_t}). \tag{2}$$

($\times_{n(t)}$ is the natural isomorphism of $\bigotimes^{n(t)} H^\star(A, R)$ with $H^\star(A^{n(t)}, R)$.)

**Recursive definition of $\tau^\star$:** For each term $\tau$, and for $N$ large enough so that $i < N$ if $x_i$ appears in $\tau$, we define $\tau^\star\colon H \longrightarrow \bigotimes^N H$ as follows.

(i) If $\tau = x_i$, then $\tau^\star = \eta_i$ (the $i^{\text{th}}$ injection of $H$ into $bigtensn(t)H$).

(ii) If $\tau = F_t(\tau_1, \ldots, \tau_n)$, then

$$\tau^\star \;=\; \tau^\bullet \circ F_t^\star, \tag{3}$$

where $\tau^\bullet\colon \bigotimes^{n(t)} H \longrightarrow \bigotimes^N H$ is defined by

$$\tau^\bullet \circ \eta_i \;=\; \tau_i^\star \tag{4}$$

for $1 \le i \le n(t)$. ($\tau_i^\star : H \longrightarrow \bigotimes^N H$ is of course available recursively. Notice that Equation (4) uniquely defines $\tau^\bullet$, by the co-product property of tensor products.)

The next lemma tells us that the recursive definition of $\tau^\star$ in (ii) is compatible with the (recursive) definition of $\overline{\tau}$. Its proof is by induction on $|\tau|$, and consists of what Gleason used to call "symbol-pushing." The relevant data for the proof are the recursive definition (page 9) of $\overline{\tau}$, Equations (3) and (4), and the functoriality of $H^\star$. We omit the details.

**Lemma 6** *For any term $\tau$, $\tau^\star \;=\; \times_N^{-1} \circ H^\star(\overline{\tau}^N)$.*

We now assume that $H = H^\star(A, R)$ has the form assumed in Theorem 4, an $R$-CGR with a single generator $a$ that lies in $H_2$ and satisfies $a^2 = 0$.

**Lemma 7** *For any continuous $\overline{F} : A^n \longrightarrow A$, there exist $\lambda \in R$ and $i \in \{1, \ldots, n\}$ such that*

$$F^\star(a) \;=\; \lambda\,(1 \otimes \cdots \otimes a \otimes \cdots \otimes 1)$$

*(with $a$ in the $i^{\text{th}}$ position and all other entries $1$).*

*Proof.* Let us define

$$e_i \;=\; 1 \otimes \cdots \otimes a \otimes \cdots \otimes 1$$

with $a$ in the $i^{\text{th}}$ position. Since $H_1 = \{0\}$, it is not hard to check that $e_i$ $(1 \leq i \leq n)$ form a basis of $(\bigotimes^n H)_2$, and so we must have

$$F^\star(a) \;=\; \sum_{i=1}^{n} \lambda_i e_i$$

for some scalars $\lambda_i \in R$. Since $a^2 = 0$, we have

$$0 \;=\; F^\star(a^2) \;=\; (F^\star(a))^2 \;=\; \sum_{i \neq j} \lambda_i \lambda_j e_i e_j \qquad (5)$$

(where the products $e_i e_i$ are obviously 0, and hence have been eliminated from this sum). It is not hard to check that the products $e_i e_j$ $(j \neq i)$ form a basis of $(\bigotimes^n H)_4$, and hence all coefficients appearing in (5) must be zero. In other words, $\lambda_i \lambda_j = 0$ for $i \neq j$. Since $R$ is an integral domain, this means that for $i \neq j$, either $\lambda_i$ or $\lambda_j$ must be zero. In other words, all but one of the $\lambda_i$ must be zero, and the conclusion of the lemma is immediate. ∎

According to Lemma 7, $\tau^\star(a)$ takes a special form, namely$\lambda \eta_i(a)$ or $\lambda e_i$. Now the general construction of $\tau^\star(a)$ depended on the arbitrary integer $N$. We have to let $N$ vary, because no single $N$ will serve for all terms $\tau$, and, ostensibly, $\lambda$ and $i$ depend on $N$. It is not terribly difficult to establish that they do not. (We omit the details.) The importance of this observation is that the automatic operations $F_t'$, which we are about to define, will thus be defined independently of $N$. Recall that the $F_t'$ are the automatic operations (i.e. constants or projections) that are needed for Theorem 4.

**Definition of the operations $F_t'$:** We now define the set $B$ and the operations $F_t'$ on $B$. In fact $B$ can be taken as any set with more than one element. We then let $c$ be any element of $B$, and define the operations $F_t'$ as follows:

(A) If $F_t^\star(a) = 0$ (i.e. if the $\lambda$ that arises in Lemma 7 is zero), then

$$F_t'(x_1, \cdots, x_{n(t)}) \;=\; c.$$

(B) If $F_t^\star(a) = \lambda e_i$, with $\lambda \neq 0$, then

$$F_t'(x_1, \cdots, x_{n(t)}) \;=\; x_i.$$

For any term $\tau$ in the operations $F_t$, we let $\tau'$ be the operation formed by replacing each $F_t$ in $\tau$ by the operation $F_t'$ defined just above. It is obvious that each $\tau'$ is either a projection operation or a constant operation with value $c$. We continue to let $a$ denote the generator of $H = H^\star(A)$.

**Lemma 8** *For any term $\tau$, if $\tau^\star(a) = 0$, then $\tau'(x) = c$ for any $x \in B^\omega$. If $\tau^\star(a) = \lambda e_i$ with $\lambda \neq 0$, then $\tau'(x) = x_i$ for any $x \in B^\omega$.*

*Proof.* By induction on $|\tau|$.

**Case 1:** $\tau = x_i$. Then $\tau^\star = \eta_i$, by part (i) of the definition of $\tau^\star$. Therefore $\tau^\star(a) = \eta_i(a) = \lambda e_i$ with $\lambda = 1$. The lemma asserts in this case that $\tau'(x) = x_i$, and indeed this is obvious.

**Case 2:** $\tau = F_t(\tau_1, \cdots, \tau_n)$. Then $F_t^\star(a) = \lambda e_i$ for some $\lambda$ and some $i$, by Lemma 7, and again, for this $i$, we have $\tau_i^\star(a) = \mu e_j$ for some $\mu$ and some $j$. And thus we have, by Equations (3) and (4),

$$\begin{aligned} \tau^\star(a) &= \tau^\bullet F_t^\star(a) \;=\; \tau^\bullet(\lambda e_i) \\ &= \lambda \tau^\bullet \eta_i(a) \;=\; \lambda \tau_i^\star(a) \;=\; \lambda \mu e_j. \end{aligned}$$

**Case 2A:** $\tau^\star(a) = 0$. This means that $\lambda\mu = 0$. Since the ring $R$ is an integral domain, either $\lambda = 0$ or $\mu = 0$. If $\lambda = 0$, then $F_t^\star(a) = 0$, and $F_t'(x) = c$ for all $x$ by (A) above. Clearly in this case $\tau'(x) = c$, as required by the lemma. On the other hand, if $\mu = 0$, then $\tau^\star(a) = 0$, and so $\tau'(a) = c$ by induction. One then easily checks that $\tau'(x) = c$, as is required for the lemma.

**Case 2B:** $\tau^\star(a) \neq 0$. This means that $\lambda\mu \neq 0$, and hence $\lambda \neq 0$ and $\mu \neq 0$. In this case we have $F_t^\star(a) \neq 0$ and $\tau_i^\star(a) \neq 0$. Thus $F_t'(x) = x_i$ by (B) above, and $\tau_i'(x) = x_j$ by induction. One then easily checks that $\tau'(x) = x_j$, as is required for the lemma. ∎

To complete the proof of Theorem 4, we consider any equation $\sigma \approx \tau$ in $\Sigma$. As we remarked in the second paragraph of the proof (page 9), we know that

$\overline{\sigma} = \overline{\tau}$. It follows easily from Lemma 6 that $\sigma^\star = \tau^\star$, and then immediately from Lemma 8 that $\sigma' = \tau'$. From there it is immediate that the automatic operations $F_t'$ model $\Sigma$. Thus $\Sigma$ is undemanding. ∎

There are many open problems, of which we will mention one. In Theorem 4 we found a way of interpreting *all* the operations on $A$ within a very simple family of operations (namely, constnts and projections). We do not know how to do this for, say, $A$ an interval.

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