1 - The Wrapping Phenomenon.

The phenomenon of **wrapping** for hypersurfaces is best illustrated by a simple example.

We consider surfaces of constant Gaussian curvature in three dimensional **hyperbolic space**. Using the **Poincaré parametrisation**, we identify this with the unit ball B in $\mathbb{R}^3$.

For any $R > 1$, the family of (Euclidian) circles of radius R passing through the north and south poles traces out a complete surface in B of constant Gaussian curvature equal to 1 (see Figure 1.1).

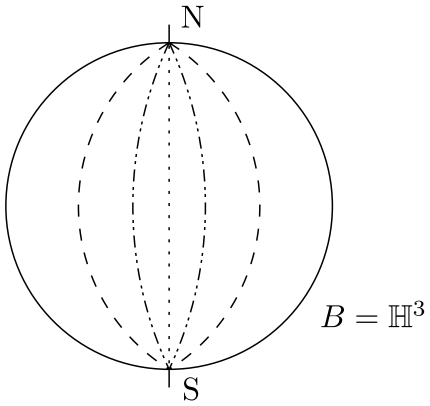

Figure 1.1

Letting R tend to $+\infty$, this family converges, in the Hausdorff sense, to the geodesic joining the north and south poles.

This illustrates the phenomenon of **wrapping**. This is a general phenomenon which describes the only way in which a sequence of convex hypersurfaces of constant curvature may degenerate. Explicitely, in the two dimensional case (see [7]):

Theorem 1.1[**Labourie**]

Let M be a complete, three dimensional manifold. Let $k > 0$ be a positive real number, and let $(\Sigma_n, p_n)_{n\in\mathbb{N}} = (S_n, i_n, p_n)_{n\in\mathbb{N}}$ be a sequence of complete, pointed immersed surfaces of **constant Gaussian curvature** *equal to k. Suppose, moreover, that $(i_n(p_n))_{n\in\mathbb{N}}$ converges to a point q_0 in M. Then:*

(i) **either** *there exists a subsequence of $(\Sigma_n, p_n)_{n\in\mathbb{N}}$ which* **converges** *to a complete, pointed immersed surface (Σ_0, p_0) of constant Gaussian curvature equal to k;*

(ii) **or** *there exists a subsequence of $(\Sigma_n, p_n)_{n\in\mathbb{N}}$ which* **degenerates** *by wrapping ever tighter round a complete geodesic.*

2 - Special Lagrangian Curvature.

The wrapping phenomenon also occurs in higher dimensions. The most appropriate setting is that of **special Lagrangian curvature**. This generalises Guassian curvature to higher dimensions, but is only defined for **convex** hypersurfaces.

Let A be a symmetric, real matrix. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of A. For $r > 0$, we define $\mathrm{SL}_r(A)$ by:

$$\mathrm{SL}_r(A) := \mathrm{Arg}(\mathrm{Det}(\mathrm{Id} + irA)) = \sum_{i=1}^{n} \arctan(r\lambda_i).$$

This function arises from the canonical special Lagangian structure of $T^*\mathbb{R}^n \cong \mathbb{C}^n$. It is an analytic function of A.

We use a function defined implicitely in terms of $SL_r(A)$. If A is positive definite, for all $\theta \in]0, n\pi/2[$, there exists a unique $r =: R_\theta(A)$ such that $\mathrm{SL}_r(A) = \theta$ (see Figure 2.2).

Definition of $R_\theta(A)$

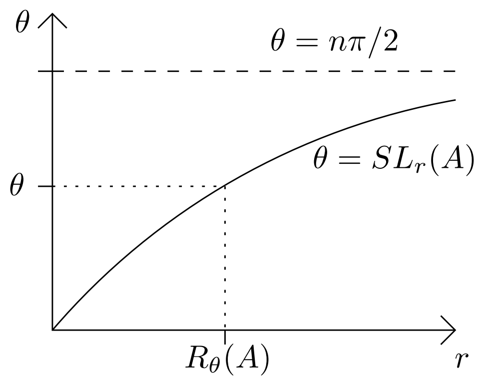

Figure 2.2

Let M be an $(n+1)$-dimensional manifold and let $\Sigma = (S, i)$ be a hypersurface immersed in M. Let A be the shape operator of Σ. For $\theta \in]0, n\pi/2[$, we define the **θ-special Lagrangian curvature** $R_\theta(\Sigma)$ of Σ by:

$$R_\theta(\Sigma) := R_\theta(A).$$

Certain special cases have a very elegant form:

(i) when $n = 2$ and $\theta = \pi/2$, $R_\theta = K^{-2}$, where K is the Gaussian curvature, and

(ii) when $n = 3$ and $\theta = \pi$, $R_\theta = (K/H)^{-2}$, where H is the mean curvature.

Importantly, **Theorem 1.1 remains true for special Lagrangian curvature** (see [9]).

3 - Applications - Quasi-Fuchsian Manifolds.

One application of hypersurfaces of constant Special Lagrangian curvature is to the study of **quasi-Fuchsian manifolds**.

Let M be a compact, n-dimensional hyperbolic manifold. We interpret $\Gamma := \pi_1(M)$ as a subgroup of $\mathrm{Isom}(\mathbb{H}^n)$.

There is a totally geodesic embedding $\hat{\imath} : \mathbb{H}^n \rightarrow \mathbb{H}^{n+1}$ which is unique up to isometries of the target space. There is a unique homomorphism $\rho : \mathrm{Isom}(\mathbb{H}^n) \rightarrow \mathrm{Isom}(\mathbb{H}^{n+1})$ with respect to which $\hat{\imath}$ is equivariant. We denote by θ_0 the restriction of ρ to Γ.

Define:

$$\hat{M}_0 = \mathbb{H}^{n+1}/\theta_0(\Gamma), \qquad i_0 : M \rightarrow \hat{M}_0 \text{ the quotient of } \hat{\imath}.$$

i_0 is a totally geodesic embedding. Trivially, $\hat{M}_0$ is homeomorphic to $M \times \mathbb{R}$. We call $\hat{M}_0$ a **Fuchsian** manifold (see Figure 3.3).

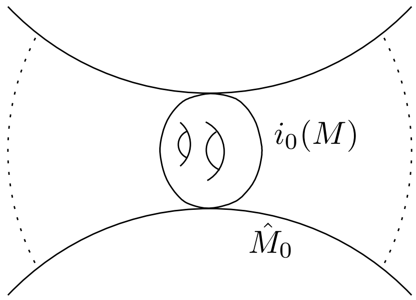

Figure 3.3

Let θ be a deformation of θ_0 satisfying certain topological conditions at ∞. Define:

$$\hat{M} = \mathbb{H}^{n+1}/\theta(\Gamma).$$

The conditions as ∞ ensure that $\hat{M}$ is still homeomorphic to $M \times \mathbb{R}$. We call $\hat{M}$ a **quasi-Fuchsian** manifold.

The **wrapping** phenomenon provides a strong compactness result. Thus, using the continuity method, we obtain (see [10]):

Theorem 3.1

Let $\hat{M}$ be a quasi-Fuchsian manifold. Let $\theta \in [(n-1)\pi/2, n\pi/2[$ be an angle, and let $r > \tan(\theta/n)$ be a positive real number. Then there exists a unique immersed hypersurface $N \subseteq \hat{M}$ in the homology class $[M \times \{0\}]$ such that:

$$R_\theta(N) = r.$$

4 - Applications - Hyperbolic Ends.

A **hyperbolic end** in the sense of Kulkarni and Pinkall is, heuristically, one half of a quasi-Fuchsian manifold.

Kulkarni and Pinkall show in [4] that **hyperbolic ends** over a compact hyperbolic manifold are in bijective equivalence with **flat conformal structures** over that manifold. This generalises the construction of Thurston when the manifold is 2 dimensional.

In certain cases, the continuity method may be used to construct unique **immersed submanifolds** of **constant special Lagrangian curvature** in the fundamental class of the hyperbolic end. In these cases, we thus obtain an equivalence between hyperbolic ends and equivariant immersed submanifolds in $\mathbb{H}^{n+1}$ of constant special Lagrangian curvature.

Since the convergence properties of these submanifolds are well understood (see Theorem 1.1, this should assist the study of moduli spaces of flat conformal structures.

5 - Bibliography.

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