CHARACTERS IN GLOBAL EQUIVARIANT HOMOTOPY THEORY

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1. Introduction

We set out from someplace quite classical: the representation theory of finite groups, in particular the theory of characters therein. Let's briefly recall how this story goes. Let G be a finite group. The goal of character theory is to understand a representation $\rho\colon G \to \mathrm{Aut}(V)$ of G on a complex vector space V via its associated *character*: the function

$$\chi_\rho := \mathrm{tr} \circ \rho\colon G \to \mathbb{C}, \tag{1.0.1}$$

where tr denotes the trace map. The beauty of character theory is that it accomplishes its goal quite thoroughly, which can be articulated succinctly as follows.

1.0.2. Definitions. (a) Consider the collection of isomorphism classes of complex representations of G, which forms a commutative semiring under the operations of direct sum and tensor product; the Grothendieck ring of this semiring, i.e. the commutative ring obtained by formally adjoining additive inverses, is denoted $\mathrm{Rep}(G)$ and referred to as the *representation ring* of G.

(b) A function on G is called a *class function* if it is conjugation-invariant. Let $\mathrm{Cl}(G;\mathbb{C})$ denote the $\mathbb{C}$-algebra of class functions $G \to \mathbb{C}$.

1.0.3. Theorem (Classical character theory). The assignment $\rho \mapsto \chi_\rho$ defined in (1.0.1) determines a ring homomorphism $\mathrm{Rep}(G) \to \mathrm{Cl}(G;\mathbb{C})$, which furthermore induces an isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{Rep}(G) \xrightarrow{\sim} \mathrm{Cl}(G;\mathbb{C})$$

of $\mathbb{C}$-algebras.

This theorem is just a restatement of the main facts one usually hears when learning character theory:

- characters behave well with respect to direct sums and tensor products of representations (because traces do);
- a representation is determined up to isomorphism by its character;
- the characters of representations span the vector space of class functions.

The aim of this thesis is to explain a generalization of this classical character theory to the setting of stable homotopy theory, due to Hopkins-Kuhn-Ravenel [5] and Stapleton [13]. This generalization might be seen as a bridge between representation theory and stable homotopy theory, but is intimately related to certain ideas in algebraic geometry as well. The confluence of these many areas of mathematics is precisely why I find this subject so fascinating, and perhaps my real aim here is just to give a sense of the many intertwining ideas at play, with (generalized) character theory as a central, motivating objective.

For now though, in this introduction, we simply seek to explain how one adapts this classical story in representation theory to a story in stable homotopy theory.

1.1. Translating

We will translate (1.0.3) into the language of homotopy theory via (topological, complex) K-theory. Recall that K-theory is a cohomology theory arising from the theory of complex vector bundles. However, for the purposes of this introduction, why don't we just focus on the degree 0 term of this cohomology theory. For a space X this is denoted $\mathrm{K}(X) = \mathrm{K}^0(X)$, and is the Grothendieck ring associated to the semiring of isomorphism classes of (complex) vector bundles on X (the operations in this semiring again coming from direct sum and tensor product). For example,

over the trivial space $X = \mathrm{pt}$, a vector bundle is simply a vector space, determined up to isomorphism by its dimension; $\mathrm{K}(\mathrm{pt})$ is therefore the Grothendieck ring of $\mathbb{Z}_{\geq 0}$, which of course is just $\mathbb{Z}$.

There are two natural *equivariant* analogues of K-theory. That is, there are two candidates which might replace K-theory when we would like to study not just spaces but spaces equipped with a G-action for some finite group G. The first is a naïve, formal construction which one can make whenever one wants a G-equivariant version of a cohomology theory, known as the associated *Borel equivariant cohomology theory*. Given a G-space X, we may always replace it with a homotopy equivalent space Y on which G acts freely: there is a contractible space $\mathrm{E}G$ with a free G action, so we may take $Y \coloneqq \mathrm{E}G \times X$. When G acts freely on Y, it is natural to expect that the G-equivariant version of a cohomology theory applied to Y recovers the original cohomology theory's value on the quotient space Y/G. This motivates the Borel construction (here applied to K-theory), given by defining

$$\mathrm{K}_G^{\mathrm{Bor}}(X) \coloneqq \mathrm{K}((\mathrm{E}G \times X)/G).$$

In particular, when X is the trivial G-space pt we have $\mathrm{K}_G^{\mathrm{Bor}}(\mathrm{pt}) \simeq \mathrm{K}(\mathrm{B}G)$, where $\mathrm{B}G \simeq \mathrm{E}G/G$ is the classifying space of G.

But for K-theory there is another, more geometric candidate for its equivariant analogue. Namely, there is a notion of a *G-equivariant (complex) vector bundle* over a G-space X. So, analagously to K-theory, we may define a G-equivariant cohomology theory which in degree 0 is given by the Grothendieck ring $\mathrm{K}_G(X)$ of G-equivariant vector bundles on X. Now, just as a vector bundle over the trivial space is just a vector space, a G-equivariant vector bundle over the trivial G-space pt is simply a representation of G. Thus we by definition have $\mathrm{K}_G(\mathrm{pt}) \simeq \mathrm{Rep}(G)$. It is with this tautological renaming of $\mathrm{Rep}(G)$ that we look to begin viewing character theory through the lens of homotopy theory.

The first non-tautological step we take is to compare these equivariant theories $\mathrm{K}_G^{\mathrm{Bor}}$ and K_G. The theory K_G still has the expected property that $\mathrm{K}_G(Y) \simeq \mathrm{K}(Y/G)$ when G acts freely on Y. Thus for any G-space X we have a natural comparison map

$$\mathrm{K}_G(X) \longrightarrow \mathrm{K}_G(\mathrm{E}G \times X) \simeq \mathrm{K}((\mathrm{E}G \times X)/G) \simeq \mathrm{K}_G^{\mathrm{Bor}}(X),$$

where the first map is induced by the projection $\mathrm{E}G \times X \longrightarrow X$. Of course this map is not an isomorphism, but a theorem of Atiyah-Segal tells us it's not terribly far from being an isomorphism. This is easiest to state for $X = \mathrm{pt}$, in which case the above gives a map[1]

$$\mathrm{Rep}(G) \simeq \mathrm{K}_G(\mathrm{pt}) \longrightarrow \mathrm{K}_G^{\mathrm{Bor}}(\mathrm{pt}) \simeq \mathrm{K}(\mathrm{B}G). \tag{1.1.1}$$

1.1.2. Definition. To a representation V of G we can associate its dimension $\dim(V)$. This extends to a ring morphism $\dim\colon \mathrm{Rep}(G) \longrightarrow \mathbb{Z}$. We call the kernel of this morphism the *augmentation ideal* of $\mathrm{Rep}(G)$, and denote it $\mathrm{Aug}(G)$.

1.1.3. Theorem (Atiyah-Segal completion). The map (1.1.1) exhibits the ring $\mathrm{K}(\mathrm{B}G)$ as the completion of the ring $\mathrm{Rep}(G)$ at the augmentation ideal $\mathrm{Aug}(G)$.

We now have a precise relationship between $\mathrm{K}(\mathrm{B}G)$ and $\mathrm{K}_G(\mathrm{pt})$, and an identification of $\mathrm{K}_G(\mathrm{pt})$ with $\mathrm{Rep}(G)$. Thus we might hope to rephrase the isomorphism of character theory (1.0.3) in terms of $\mathrm{K}(\mathrm{B}G)$. We take one further step in order to accomplish this.

[1] This map can alternatively be described as follows: a representation of G is a map $\pi_1(\mathrm{B}G) \simeq G \longrightarrow \mathrm{Aut}(V)$, which determines a local system with fiber V on $\mathrm{B}G$, which determines a vector bundle with fiber V on $\mathrm{B}G$.

1.1.4. Notation. For the remainder of this subsection and the next, fix a prime p.

1.1.5. Lemma. Suppose G is a p-group. Then for some $n \in \mathbb{Z}_{>0}$, the ideal $\mathrm{Aug}(G)^n$ is contained in the ideal $p \cdot \mathrm{Rep}(G)$.

The proof is relegated to a footnote.[2]

1.1.6. Lemma. If G is a p-group, the Atiyah-Segal map (1.1.1) is an isomorphism after p-completion. So $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Rep}(G) \simeq \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{K}(\mathrm{B}G)$.

Proof. This follows immediately from (1.1.3, 1.1.5), the latter implying that completing $\mathrm{Rep}(G)$ at $\mathrm{Aug}(G)$ is subsumed by completing at p. That is, completing at $\mathrm{Aug}(G)$ and then completing at p is equivalent to just completing at p. □

This is nice. If we pick an embedding $\mathbb{Z}_p \to \mathbb{C}$, we can now rewrite one side of character theory (1.0.3) completely in terms of ordinary K-theory when G is a p-group:

$$\mathbb{C} \otimes_{\mathbb{Z}} \mathrm{Rep}(G) \simeq \mathbb{C} \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Rep}(G)) \simeq \mathbb{C} \otimes_{\mathbb{Z}_p} \widehat{\mathrm{K}}_p(\mathrm{B}G),$$

where $\widehat{\mathrm{K}}_p(\mathrm{B}G) := \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{K}(\mathrm{B}G)$ is the p-completion of the K-theory of $\mathrm{B}G$.

The other side of character theory can be rewritten in terms of the classifying space $\mathrm{B}G$ as well. If $\Omega\mathrm{B}G := \mathrm{Map}_*(\mathrm{S}^1, \mathrm{B}G)$ denotes the pointed loop space, then its connected components are given by

$$\pi_0(\Omega\mathrm{B}G) \simeq \pi_1(\mathrm{B}G) \simeq G.$$

So the singular cohomology with complex coefficients $\mathrm{H}^0(\Omega\mathrm{B}G; \mathbb{C})$ in degree 0 is just the vector space of functions $G \to \mathbb{C}$. Intuitively, changing the basepoint in $\mathrm{B}G$ corresponds to conjugation in G, and indeed it's not hard to show that if we instead take the unpointed, or *free*, loop space $\mathscr{L}\mathrm{B}G := \mathrm{Map}(\mathrm{S}^1, \mathrm{B}G)$, then its connected components are given by G modulo conjugation. And thus

$$\mathrm{H}^0(\mathscr{L}\mathrm{B}G; \mathbb{C}) \simeq \mathrm{Cl}(G; \mathbb{C}).$$

So finally we have completely translated character theory into the language of homotopy theory: for G a p-group we have an isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}_p} \widehat{\mathrm{K}}_p(\mathrm{B}G) \simeq \mathrm{H}^0(\mathscr{L}\mathrm{B}G; \mathbb{C}). \tag{1.1.7}$$

[2] I quite like this proof, but couldn't justify interrupting our story with a little fact from representation theory. In any case, it's still here for the curious:

Proof of (1.1.5). The ideal $\mathrm{Aug}(G)$ is additively generated by elements of the form $[V] - d$, where V is an irreducible representation of G and $d := \dim(V)$ is the class of the $\dim(V)$-dimensional trivial representation of G. Since G has only finitely many irreducible representations V, it suffices to show that, fixing such a V, the element $([V] - d)^n$ is in the ideal $p \cdot \mathrm{Rep}(G)$ for some $n \in \mathbb{Z}_{>0}$.

Now, Brauer's induction theorem tells us that $[V] - d$ is a linear combination of elements of the form $\mathrm{Ind}_H^G([c] - 1)$ where $H \subseteq G$ is a subgroup, c is a one-dimensional representation (i.e. linear character) on H, and Ind_H^G denotes induction from H to G. So it now suffices to show for any such H, c that $\mathrm{Ind}_H^G([c] - 1)^n \in p \cdot \mathrm{Rep}(G)$ for some $n \in \mathbb{Z}_{>0}$. We now have two cases:

- Suppose H is a proper subgroup of G. The push-pull formula tells us that

  $$\mathrm{Ind}_H^G([c] - 1)^n = \mathrm{Ind}_H^G\left(([c] - 1) \cdot (\mathrm{Ind}_H^G([c] - 1)|_H)^{n-1}\right)$$

  where $|_H$ denotes restriction from G to H. By induction (the other kind now) on the order of our group, $(\mathrm{Ind}_H^G([c] - 1)|_H)^{n-1} \in p \cdot \mathrm{Rep}(H)$ for some $n \in \mathbb{Z}_{>0}$, implying $\mathrm{Ind}_H^G([c] - 1)^n \in p \cdot \mathrm{Rep}(G)$, as desired.
- Else $H = G$, in which case G being a p-group implies $[c]^{p^m} = 1$ for some $m \in \mathbb{Z}_{>0}$. But then the binomial theorem implies $[c - 1]^{p^m} \in p \cdot \mathrm{Rep}(G)$, as desired. □

1.2. Generalizing

We now seek to describe the generalizations of (1.1.7) that were found by Hopkins-Kuhn-Ravenel and Stapleton, and which will be proved in this thesis. There are many new notions which need to be introduced to state these generalizations, of which we can only give a vague summary here. Indeed, a significant portion of this thesis is devoted to introducing and developing these notions, and the truly interested reader is kindly invited to continue reading after the nebulous haze of intuition which follows. Nevertheless, we must complete the story we set out to tell in this introduction; hopefully the following step-by-step path of generalization from (1.1.7) to the main theorem (1.2.6) serves as a coherent and useful way to do so.

- By replacing the free loop space functor with a sort of p-adic loop space functor $\mathscr{L}$, we may obtain the isomorphism (1.1.7) not only for p-groups, but for all finite groups G.
- Instead of just p-adically completing the group $\mathrm{K}(\mathrm{B}G)$, one can p-adically complete K-theory as a cohomology theory, and thus obtain a full cohomology theory $\widehat{\mathrm{K}}_p$. One can then upgrade (1.1.7) into an isomorphism in all degrees by replacing the right-hand side with a periodic version of singular cohomology:

$$\mathbb{C} \otimes_{\mathbb{Z}_p} \widehat{\mathrm{K}}_p^*(\mathrm{B}G) \simeq \prod_{k \in \mathbb{Z}} \mathrm{H}^{2k+*}(\mathscr{L}\mathrm{B}G; \mathbb{C}). \tag{1.2.1}$$

- Given a cohomology theory E, there is a process of obtaining a "rationalized cohomology theory", denoted $\mathbb{Q} \otimes E$; as the notation suggests, this is analogous to the operation of tensoring abelian groups with $\mathbb{Q}$. In the case of K-theory, rationalizing simply leaves us with periodic rational singular cohomology.[3] Stated more precisely for p-adically completed K-theory, there is a natural isomorphism of cohomology theories

$$(\mathbb{Q} \otimes \widehat{\mathrm{K}}_p)^*(-) \simeq \prod_{k \in \mathbb{Z}} \mathrm{H}^{2k+*}(-; \mathbb{Q}_p).$$

  Therefore, we may restate (1.2.1) as

$$\mathbb{C} \otimes_{\mathbb{Z}_p} \widehat{\mathrm{K}}_p^*(\mathrm{B}G) \simeq \mathbb{C} \otimes_{\mathbb{Q}_p} (\mathbb{Q} \otimes \widehat{\mathrm{K}}_p)^*(\mathscr{L}\mathrm{B}G). \tag{1.2.2}$$

- It is not in fact necessary to extend coefficients all the way to $\mathbb{C}$. One may recall that even in the statement of classical character theory (1.0.3), it suffices to extend only to a field extension of $\mathbb{Q}$ containing all of the roots of unity. Similarly, in (1.2.2) we may replace $\mathbb{C}$ with the maximal ramified extension $\operatorname{colim}_k \mathbb{Q}_p(\zeta_{p^k})$ of $\mathbb{Q}_p$.
- The p-adic completion of K-theory is the first member of a naturally occurring family of cohomology theories $\{E(n)\}_{n \geq 0}$, known as the Morava E-theories. We will show that character theory generalizes to these cohomology theories as follows. For a cohomology theory F, let F^* denote its coefficient ring, i.e. its value on a point, $F^*(\mathrm{pt})$. Then, for $E := E(n)$, there is a nonzero ring extension C_0^* of $\mathbb{Q} \otimes E^*$ such that

$$C_0^* \otimes_{E^*} E^*(\mathrm{B}G) \simeq C_0^* \otimes_{\mathbb{Q} \otimes E^*} (\mathbb{Q} \otimes E)^*(\mathscr{L}^n \mathrm{B}G). \tag{1.2.3}$$

  where $\mathscr{L}^n$ denotes the n-fold composition of the (p-adic) loop space functor.
- Again in analogy to the situation for abelian groups, we should think of rationalization as some kind of localization process. In fact, rationalization is the zeroth member of a naturally occurring family of localization processes on cohomology theories $\{L_{K(t)}\}_{t \geq 0}$. We will prove more generally that for $E := E(n)$ and

[3] The phrase "Chern character" is relevant here.

$L_t := L_{K(t)}E$ with $0 \leq t < n$, there is a nonzero extension C_t^* of L_t^* (which, as in the case of rationalization, is itself an extension of E^*) such that

$$C_t^* \otimes_{E^*} E^*(\mathrm{B}G) \simeq C_t^* \otimes_{L_t^*} L_t^*(\mathscr{L}^{n-t}\mathrm{B}G). \tag{1.2.4}$$

- In fact, (1.2.4) is just one instance of a natural isomorphism of G-equivariant cohomology theories, arising from the Borel equivariant cohomology theories associated to E and L_t. On finite G-CW complexes X, this isomorphism looks like:

$$C_t^* \otimes_{E_t^*} E^*(\mathrm{E}G \times_G X) \simeq C_t^* \otimes_{L_t^*} L_t^*(\mathrm{E}G \times_G \mathscr{L}^{n-t}X). \tag{1.2.5}$$

- Finally, observe that (1.2.5) is a statement about G-equivariant cohomology theories which doesn't seem biased in any way toward what group G we're working with. In fact, we can think of the two sides of this isomorphism as cohomology theories which are in some sense "equivariant with respect to all finite groups G". We call such things *global equivariant cohomology theories*. Our proof of these statements will critically use this "global" perspective, and setting up the framework to study these global equivariant cohomology theories is one our primary goals.

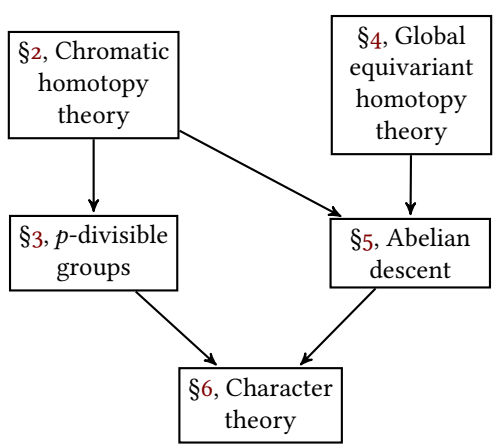
So finally, the result we will be working towards can be stated as follows.

1.2.6. Theorem (Informal). There is an isomorphism (1.2.5), where each side may be viewed as a global equivariant cohomology theory over all finite groups G.

1.2.7. Remark. We should explicitly note here that this thesis is almost entirely expository. However, our strong commitment to the global perspective in this exposition is in some sense original, and seems quite interesting and useful. In any case, thanks to Hopkins-Kuhn-Ravenel and Stapleton for producing some incredibly interesting mathematics.

1.3. Overview

The remainder of our work is organized as follows. In §2 we review some of the main ideas from the field of *chromatic homotopy theory*, which is where the Morava E-theories arise. In §3 we discuss how the theory of *p-divisible groups* in algebraic geometry appears in chromatic homotopy theory, which appearance is central to understanding character theory. In §4 we set up our framework for studying global equivariant cohomology theories and the like. In §5 we study a particular property of global equivariant cohomology theories which is key to our proof of character theory. Finally in §6 we actually give a formal statement and proof of our generalization of character theory. The following graph depicts the dependency of these sections on one another, in case the reader is interested in just a portion of this thesis.



1.4. Mathematical and notational conventions

I have tried throughout the text to give introductions to the many background ideas relevant in this thesis, so that even readers unfamiliar with some of these ideas might get something out of reading (parts of) it. However, it was of course necessary to assume some foundational language, so we say a word about some of our higher-level prerequisites here:

- I will freely use the basic language of stable homotopy theory, most notably the notions of spectra, ring spectra, and E_∞-ring spectra. The reader unfamiliar with these terms should just replace the first two terms with cohomology theories and multiplicative cohomology theories, respectively. The last should be thought of as really nice multiplicative cohomology theories, which have an associated theory analagous to commutative rings in algebra. E.g. there is a good notion of modules of E_∞-rings and tensor products of these modules, and when one encounters such notions in the text, one should just imagine them as the correct counterpart in stable homotopy theory of the usual notions in algebra.
- I will freely use higher category theory, in particular the theory of ∞-categories. However, the reader is strongly urged not to worry about this, as long as they are familiar with ordinary category theory. The theory will be treated completely as a black-box, and used formally, analagously to ordinary category theory. I will explicitly state when we are viewing something as a higher category, but when working with it I will not distinguish the associated higher-categorical notions notationally or terminologically from ordinary categorical notions. For example, if we are dealing with an ∞-category, then all limits and colimits refer to the correct ∞-categorical notions, i.e. homotopy limits and colimits, but will still just be denoted lim and colim. The same goes for all other categorical notions: subcategories, functors, adjunctions, Kan extensions, and so on.

And finally we state a couple of conventions that will be employed throughout:

- All rings and algebras will be commutative or graded-commutative, whichever the context makes more natural.
- The ∞-category of spaces (i.e. topological spaces or Kan complexes up to weak equivalence) is denoted Space. The ∞-category of spectra is denoted Spect.

2. Chromatic homotopy theory

In this section we lay out some of the fundamental ideas in chromatic homotopy theory, enough that we can somewhat safely speak about the generalizations of character theory alluded to in §1. As this section is intended primarily as background, it's essentially void of proofs. The lecture notes [4, 7] are nice places to read about these ideas in more depth and detail (and my debt to these sources in this exposition will be clear).

2.1. Formal group laws & Chern classes

Our starting point is the theory of Chern classes associated to complex vector bundles. Actually, let's just focus on (complex) line bundles $L \to X$, in which case all that's of interest is the first Chern class $c_1(L) \in \mathrm{H}^2(X) := \mathrm{H}^2(X; \mathbb{Z})$. Recall two facts about this situation:

- Chern classes are natural: the Chern class $c_1(f^*L)$ of the pullback of a line bundle $L \to X$ in a map $f\colon Y \to X$ is the pullback $f^*(c_1(L)) \in \mathrm{H}^2(Y)$ of the Chern class of L.

- There is a universal line bundle: letting $\mathcal{O}(1)$ denote the tautological line bundle over the space $\mathbb{CP}^\infty \simeq \mathrm{BU}(1)$, there is a natural bijection between homotopy classes of maps $f\colon X \to \mathbb{CP}^\infty$ and isomorphism classes of line bundles on X, given by associating to such a map f the pullback $f^*(\mathcal{O}(1))$ of the tautological bundle. That is, the functor associating to a space X the set of isomorphism classes of line bundles on it is representable (in the homotopy category of spaces) by $\mathbb{CP}^\infty$, and the tautological bundle $\mathcal{O}(1)$ is truly tautological: it corresponds to the identity map $\mathrm{id}\colon \mathbb{CP}^\infty \to \mathbb{CP}^\infty$.

Of course Chern classes are also isomorphism-invariant, so it follows from these facts that the Chern classes of *any* line bundles L is determined by a choice of Chern class $c_1(\mathcal{O}(1)) \in \mathrm{H}^2(\mathbb{CP}^\infty)$. Now, recall that the cohomology ring of $\mathbb{CP}^\infty$ is given by[4]

$$\mathrm{H}^*(\mathbb{CP}^\infty) \simeq \mathbb{Z}[\![t]\!],$$

where the generator lives in degree 2, i.e. $t \in \mathrm{H}^2(\mathbb{CP}^\infty)$. So of course then the interesting choice to make is $c_1(\mathcal{O}(1)) := t$. But note that t is really only well-defined up to a unit in $\mathbb{Z}$, i.e. up to sign! So to be precise we need to fix such a generator and then declare this to be $c_1(\mathcal{O}(1))$.

Now, the set of isomorphism classes of line bundles on a space X is often referred to as its Picard group; the name isn't so important right now, but it is important that it's a group. Recall that the operation is given by tensor product of line bundles. So it's natural to ask: given two line bundles L and L' on X, can we express $c_1(L \otimes L')$ in terms of $c_1(L)$ and $c_1(L')$? In fact one can do so quite easily:

$$c_1(L \otimes L') = c_1(L) + c_1(L') \in \mathrm{H}^2(X). \tag{2.1.1}$$

Since there is universal line bundle $\mathcal{O}(1)$, proving this boils down to proving it for the universal example. The universal *pair* of line bundles is given by the bundles $\pi_1^*\mathcal{O}(1), \pi_2^*\mathcal{O}(1)$ on $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$, where $\pi_1, \pi_2\colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ denote the projections. So the formula (2.1.1) holds if and only if it holds for this universal example, that is if

$$c_1(\pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1)) = c_1(\pi_1^*\mathcal{O}(1)) + c_1(\pi_2^*\mathcal{O}(1)) \in \mathrm{H}^2(\mathbb{CP}^\infty \times \mathbb{CP}^\infty).$$

To see that this is true, we observe that the canonical map $\iota\colon \mathbb{CP}^\infty \vee \mathbb{CP}^\infty \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ induces an isomorphism in H^2,

$$\iota^*\colon \mathrm{H}^2(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \xrightarrow{\sim} \mathrm{H}^2(\mathbb{CP}^\infty \vee \mathbb{CP}^\infty), \tag{2.1.2}$$

and that the pullback bundle $\iota^*(\pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1))$ is just a copy of $\mathcal{O}(1)$ over each summand of $\mathbb{CP}^\infty \vee \mathbb{CP}^\infty$.

We can alternatively view this universal example as follows: by the Yoneda lemma, the tensor product operation on line bundles corresponds to some kind of multiplication map $\mu\colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$. In cohomology this gives a map

$$\mathbb{Z}[\![t]\!] \simeq \mathrm{H}^*(\mathbb{CP}^\infty) \xrightarrow{\mu^*} \mathrm{H}^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq \mathbb{Z}[\![x, y]\!],$$

[4]It may be more common to write $\mathrm{H}^*(\mathbb{CP}^\infty) \simeq \mathbb{Z}[t]$, i.e. to use the polynomial ring rather than the power series ring. This is just a convention: does one want to assemble the graded pieces of $\mathrm{H}^*(X)$ by taking their direct product or their direct sum? In other words, does one want to consider infinite or just finite sums of homogenous elements in the ring $\mathrm{H}^*(X)$? In just a bit we'll be considering cohomology theories E other than singular cohomology, in particular ones in which the coefficient ring $E^* := E^*(\mathrm{pt})$ is not concentrated in degree 0, but may in fact be nonzero in infinitely many degrees. But even in this more general situation, we'd like to think of E^* as some sort of base ring of coefficients, in which case we may have infinite sums in the ring $E^*(X)$ still living in some finite graded piece. It will thus be more natural to consider the power series ring $E^*[\![t]\!]$, and so we begin with this convention here.

using the Kunneth isomorphism on the right-hand side, so $x = \pi_1^*(t)$ and $y = \pi_2^*(t)$. By definition we'll have $\mu^*\mathcal{O}(1) \simeq \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1)$. So we end up with

$$\mu^*(t) = \mu^*(c_1(\mathcal{O}(1))) = c_1(\mu^*\mathcal{O}(1)) = c_1(\pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1))$$

and

$$\begin{aligned} x + y = \pi_1^*(t) + \pi_2^*(t) &= \pi_1^*(c_1(\mathcal{O}(1))) + \pi_2^*(c_1(\mathcal{O}(1))) \\ &= c_1(\pi_1^*\mathcal{O}(1)) + c_1(\pi_2^*\mathcal{O}(1)). \end{aligned}$$

Therefore we conclude that the formula (2.1.1) is equivalent to the formula $\mu^*(t) = x+y$.

Somebody interested in stable homotopy theory might now ask: what precisely did we use about singular cohomology in the above? We observe that the essential facts were the computations of $\mathrm{H}^*(\mathbb{CP}^\infty)$ and $\mathrm{H}^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$ as power series rings. This motivates the following definition.[5]

2.1.3. Definition. A multiplicative cohomology theory, i.e. a (homotopy) ring spectrum, E is said to be *complex-orientable* if the Atiyah-Hirzebruch spectral sequence

$$\mathrm{H}^p(\mathbb{CP}^\infty; E^q(\mathrm{pt})) \Rightarrow E^{p+q}(\mathbb{CP}^\infty)$$

degenerates at its second page. By the standard computations in singular cohomology, this condition implies that there is an isomorphism $E^*(\mathbb{CP}^\infty) \simeq E^*[\![t]\!]$, where $E^* := E^*(\mathrm{pt})$. A choice of such an isomorphism, i.e. a choice of generator t (which again is well-defined only up to the units of E^0) is referred to as a *complex-orientation* of E. We say E is *complex-oriented* if it is equipped with a complex-orientation.

In fact the degeneration of the spectral sequence for $\mathbb{CP}^\infty$ forces the analagous degeneration for $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$, whence a complex-orientation on E similarly determines an isomorphism $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq E^*[\![x, y]\!]$.

2.1.4. Examples. We give the two most basic examples of complex-orientable cohomology theories, which are essentially the two motivating examples of cohomology theories in the first place:

(a) Obviously singular cohomology H is complex-orientable.
(b) One can show that complex K-theory K is complex-orientable as well.

So we can create a theory of Chern classes with values in any complex-oriented cohomology theory E. But now there is no reason for formula (2.1.1) to hold in general. Its validity in singular cohomology relies on the fact (2.1.2), which relies on the coefficient ring $\mathrm{H}^* \simeq \mathbb{Z}$ being concentrated in degree 0; this of course is not true in general, e.g. for complex K-theory, Bott periodicity tells us $\mathrm{K}^* \simeq \mathbb{Z}[\beta, \beta^{-1}]$ with β in degree -2. However, we can still say something about the Chern class of a tensor product with values in E^*: it is still determined by the universal example, which is still determined by the map in cohomology

$$E^*[\![t]\!] \simeq E^*(\mathbb{CP}^\infty) \xrightarrow{\mu^*} E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq E^*[\![x, y]\!], \tag{2.1.5}$$

and again the formula for the Chern class of a tensor product will be given by the power series $f(x, y) := \mu^*(t) \in E^*[\![x, y]\!]$. The fact that the tensor product of line bundles is unital, commutative, and associative (all up to isomorphism) implies that f is a special kind of power series in two variables, which we can think of as an infinitesimal version of a group operation.

[5] One can also give alternative, somewhat more elementary (but equivalent) definitions of complex-orientability than the one given here, but this is the one which most easily and succinctly fits into the motivation being given here.

2.1.6. Definition. A *formal group law* over a commutative ring R is a power series $f \in R[\![x, y]\!]$ satisfying:

(a) $f(x, 0) = x$ and $f(0, y) = y$;

(b) $f(x, y) = f(y, x)$;

(c) $f(f(x, y), z) = f(x, f(y, z))$.

2.1.7. Remark. Note that we'll always speak about formal group laws over *commutative* rings, but the coefficient ring E^* of a multiplicative cohomology theory is *graded-commutative*. This is no problem here: since μ^* respects grading and the classes t, x, y in (2.1.5) are in degree 2, the power series $f := \mu^*(t)$ must have coefficients in the even degrees E^{2*}, which of course do form a commutative ring.

2.1.8. Examples. We give the two most basic examples (over any commutative ring R):

(a) First we have the *additive formal group law*, $f(x, y) = x + y$. We saw above that this arises in stable homotopy theory via singular cohomology.

(b) Second we have the *multiplicative formal group law*, $f(x, y) = x + y + xy = (1 + x)(1 + y) - 1$. One should think of this really as multiplication xy, except with the identity shifted to 0 rather than 1. One can show without much trouble that this arises as the formal group law associated to complex K-theory.

It turns out that this assignment

$$\{\text{complex-oriented cohomology theories}\} \longrightarrow \{\text{formal group laws}\}$$

defined by (2.1.5) is an absurdly interesting one to consider. Maybe one is already astounded by the fact that the examples in (2.1.4) and (2.1.8) are entirely parallel, i.e. the two simplest formal group laws correspond to the two simplest (in some sense, maybe historical or pedagogical) cohomology theories; I certainly still am! This construction is some sort of approximation of topological objects—cohomology theories, i.e. spectra—by algebraic objects—formal group laws. This is precisely the sort of machine that algebraic topology is all about, but this is a case of the algebraic approximation retaining a miraculous amount of information about the topology. Chromatic homotopy theory essentially refers to the business of understanding just how good this approximation is, and we'll review some of the key aspects of this theory in the following two subsections. Just as this one is, the next two subsections have titles of the form "A & B"; in all three cases A is some concept in the theory of formal groups and B is the avatar of A in stable homotopy theory.

2.2. The Lazard ring & complex bordism

2.2.1. The first important observation to make about formal group laws is that there's a universal one. A formal group law is a power series

$$f(x, y) = \sum_{i,j} a_{i,j} x^i y^j$$

with coefficients in a ring R, satisfiying three conditions. The power series is of course formally determined by the coefficients $a_{i,j}$, and the conditions can be expressed purely in terms of the coefficients as well:

(a) that $f(x, 0) = x$ is equivalent to $a_{i,0}$ being 1 for $i = 1$ and 0 otherwise, and similarly for $f(0, y) = y$;

(b) that $f(x, y) = f(y, x)$ is equivalent to $a_{i,j} = a_{j,i}$;

(c) that $f(x, f(y, z)) = f(f(x, y), z)$ is again equivalent to certain integer polynomial relations among the coefficients $a_{i,j}$, but these are more complicated and omitted here.

We conclude that there is some ideal $I \subseteq \mathbb{Z}[a_{i,j}]$ such that specifying a formal group law over a ring R is equivalent to specifying a morphism of rings $\mathbb{Z}[a_{i,j}]/I \to R$. That is, the formal group law $\sum_{i,j} a_{i,j} x^i y^j$ over the ring $\mathrm{L} := \mathbb{Z}[a_{i,j}]/I$ is the universal example of a formal group law. For example, we can rephrase the discussion of the previous section by saying that a complex-orientation on a ring spectrum E determines a morphism of rings $\mathrm{L} \to E^*$. In fact, there is a natural (even) grading on L so that this is a morphism of graded rings. The (graded) ring L is known as the *Lazard ring*.

2.2.2. It turns out that there is also a universal example of a complex-oriented cohomology theory, known as *complex bordism* and denoted MU. (As the name suggests, MU is connected to (co)bordisms of manifolds, but we have no time to discuss its origins or construction.) More precisely, there is a bijection between complex-orientations of a ring spectrum E and homotopy classes of maps of ring spectra $\mathrm{MU} \to E$. In particular, there is a *canonical* complex-orientation of MU, corresponding to the identity map $\mathrm{id}\colon \mathrm{MU} \to \mathrm{MU}$.

We have now given two universal examples: a complex-orientation of a ring spectrum E canonically determines morphisms $\mathrm{MU} \to E$ and $\mathrm{L} \to E^*$. It is hopefully natural to guess the following result then.

2.2.3. Theorem (Quillen). The map $L \to \mathrm{MU}^*$ determined by the canonical complex-orientation of MU is an isomorphism. In other words, the formal group law associated to MU is the universal one, and the map $\mathrm{L} \simeq \mathrm{MU}^* \to E^*$ induced by a complex-orientation $\mathrm{MU} \to E$ of a ring spectrum E is precisely the map classifying the formal group law on E^*.

2.2.4. Remark. Note that we said it was natural to guess the previous result, but then labeled it a theorem and attributed it to Quillen. Indeed, the result isn't formal or easy, but an incredibly deep computation, and sort of miraculous! Again, we have no time here to delve into the details.

One reason to get really excited about (2.2.3) is that it inspires a method for inverting the process of extracting formal group laws from complex-oriented cohomology theories. That is, if we are given a formal group law over some ring R, classified by a map $\mathrm{MU}^* \simeq \mathrm{L} \to R$, we could try to define a complex-oriented cohomology theory E with exactly this associated formal group law by defining

$$E^*(X) := \mathrm{MU}^*(X) \otimes_{\mathrm{L}} R$$

(for finite CW complexes X). But of course this won't always define a cohomology theory. A priori, to retain the necessary exact sequences we would need to assume the map $\mathrm{L} \to R$ is flat. However, this is quite a stringent condition. Another important theorem tells us that the Lazard ring L is isomorphic to an infinite polynomial ring $\mathbb{Z}[b_1, b_2, \ldots]$, and flatness over such a large ring is indeed a rather limiting hypothesis (maybe this is easier to understand geometrically: imagine trying to be flat over an infinite-dimensional affine space!).

Luckily, it turns out that we can get away with a significantly weaker flatness hypothesis to obtain a cohomology theory. To state this correctly we should introduce formal groups, the more invariant, coordinate-free object underlying formal group laws.

2.2.5. Notation. If R is a ring, let Alg_R denote the category of R-algebras.

Suppose we are given a formal group law $f \in R[\![x, y]\!]$. We are supposed to think of this as some kind of group operation, but it's a power series so it doesn't quite make sense to actually apply the operation to elements of R. However, we can apply it to nilpotent elements of R. More precisely, if $\mathrm{Nil}\colon \mathrm{Alg}_R \to \mathrm{Set}$ denotes the functor sending an R-algebra to its set of nilpotent elements, then f defines a lift of Nil to a functor to abelian groups, $\mathbb{G}_f\colon \mathrm{Alg}_R \to \mathrm{Ab}$. Note that Nil is the colimit of the functors corepresented by the R-algebras $R[t]/(t^n)$. So, algebro-geometrically Nil just corresponds to the formal scheme $\mathrm{Spf}(R[\![t]\!]) \simeq \mathrm{colim}_n \mathrm{Spec}(R[t]/(t^n))$, and the formal group law f determines a group sctructure on this formal scheme. More generally, we make the following definition.

2.2.6. Definition. A *formal group* over a ring R is a functor $\mathbb{G}\colon \mathrm{Alg}_R \to \mathrm{Ab}$ which is:

(a) a sheaf with respect to the Zariski topology on Alg_R;

(b) Zariski-locally isomorphic to functors of the form $\mathbb{G}_f$, for f a formal group law.

Perhaps this definition is a bit opaque, so let's elaborate for a little while. We said that formal groups are a more invariant notion than formal group laws, but how exactly? In other words, given two formal group laws f, f' over a ring R, when are the associated formal groups $\mathbb{G}_f, \mathbb{G}_{f'}$ isomorphic? One can show that the formal groups are isomorphic precisely when there is a "change-of-variable" relating f and f', i.e. an invertible power series $g \in R[\![t]\!]$ such that $f(g(x), g(y)) = g(f'(x, y))$. So we can think alternatively think of a formal group over R as the following data:

- an open covering of R, i.e. elements $r_1, \ldots, r_n \in R$ such that $(r_1, \ldots, r_n) = (1) = R$;
- formal group laws f_i over each localization $R[r_i^{-1}]$;
- changes-of-variable $g_{i,j} \in R[(r_i r_j)^{-1}][\![t]\!]$ relating f_i and f_j, i.e. such that
$$g_{i,j} f_i(g_{i,j}^{-1}(x), g_{i,j}^{-1}(y)) = f_j(x, y) \in R[(r_i r_j)^{-1}][\![x, y]\!]$$
- which are coherent in that there are identifications $g_{i,k} = g_{j,k} \circ g_{i,j}$ over $R[(r_i r_j r_k)^{-1}][\![t]\!]$.

This is just the natural way to glue together formal group laws along changes-of-variable. This can also be thought of naturally in the language of stacks. We have seen that there is a "moduli stack of formal group laws" $\mathscr{M}_{\mathrm{FGL}}$, i.e. an algebro-geometric object such that maps $\mathrm{Spec}(R) \to \mathscr{M}_{\mathrm{FGL}}$ are in bijection with formal group laws over R: it is just the affine scheme Spec(L) associated to the Lazard ring. Then change-of-variable can be encoded as the action of a group scheme G (paramterizing invertible power series $g(t) = b_1 t + b_2 t^2 + \cdots$) on the scheme $\mathrm{Spec}(L)$ (paramterizing formal group laws), the action sending a formal group law $f(x, y)$ to the formal group law $g(f(g^{-1}(x), g^{-1}(y))$. Hence the "moduli stack of formal groups" should be defined to be the quotient stack $\mathscr{M}_{\mathrm{FG}} := \mathrm{Spec}(\mathrm{L})/G$. And indeed when one unwraps all of the definitions, one now has a groupoid of maps $\mathrm{Spec}(R) \to \mathscr{M}_{\mathrm{FG}}$, which is precisely the groupoid of formal groups over R, as defined above. In this language, the quotient map $\mathrm{Spec}(\mathrm{L}) \to \mathscr{M}_{\mathrm{FG}}$ corresponds to taking the formal group underlying a formal group law.

Let us now return to the problem of defining a cohomology theory $E^*(X) := \mathrm{MU}^*(X) \otimes_{\mathrm{L}} R$ given a formal group law over R. As we said above, one might expect that we need flatness of $\mathrm{L} \to R$, i.e. of $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathrm{L})$, to do so. But in fact one can show that it suffices that the composite $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathrm{L}) \to \mathscr{M}_{\mathrm{FG}}$ be flat. This motivates the following definition.

2.2.7. Definition. We say a formal group over a ring R is *Landweber-exact* if the classifying map $\mathrm{Spec}(R) \to \mathscr{M}_{\mathrm{FG}}$ is flat. We say a formal group law over R is Landweber-exact if its underlying formal group is.

2.2.8. Remark. The terminology of Landweber-exactness originates from a (very useful) theorem of Landweber, the *Landweber exact functor theorem*, which gives a simple algebraic (necessary and sufficient) criterion for a formal group law to be Landweber-exact.

With more work, one can refine our two constructions extracting formal group laws from complex-oriented cohomology theories and building complex-oriented cohomology theories from Landweber exact formal group laws as follows.

2.2.9. Definition. We say a ring spectrum E is *weakly even periodic* if

(a) $E^i \simeq 0$ for i odd;

(b) the multiplication map $E^2 \otimes_{E^0} E^{-2} \to E^0$ is an isomorphism.

The second condition implies that E^2 is an invertible E^0-module and that $E^{2k} \simeq (E^2)^{\otimes k}$ for $k \in \mathbb{Z}$. We say E is *even periodic* if it satisfies the following stronger condition:

(b′) there is an invertible element $\beta \in E^{-2}$, so that multiplication by β determines an isomorphism $E^k \simeq E^{k-2}$ for all $k \in \mathbb{Z}$.

2.2.10. Remark. Any weakly even periodic ring spectrum E is complex-orientable. If E is even periodic, then it's easy to see that the associated formal group law over E^* can be viewed simply as a formal group law over E^0. In fact, even if E is just weakly even periodic, there is a formal group over E^0, i.e. a map $\mathrm{Spec}(E^0) \to \mathscr{M}_{\mathrm{FG}}$, which when pulled back to E^* is just the formal group $\mathrm{Spec}(E^*) \to \mathrm{Spec}(E^0) \to \mathscr{M}_{\mathrm{FG}}$ underlying the formal group law over E^*.

2.2.11. Proposition. Consider the category of Landweber-exact formal groups, i.e. the category of flat maps $\mathbb{G}\colon \mathrm{Spec}(R) \to \mathscr{M}_{\mathrm{FG}}$. There is a functor $\mathbb{G} \mapsto E_{\mathbb{G}}$ from this category to the homotopy category of weakly even periodic ring spectra. Moreover, this functor is an equivalence of categories, with the inverse functor sending E to the formal group over E^0 discussed in (2.2.10).

2.3. Stratifying by height & Morava K- and E-theories

In this subsection, we strengthen even further the intimate relationship between stable homotopy theory and formal groups by discussing some geometric structure of the moduli stack of formal groups $\mathscr{M}_{\mathrm{FG}}$, and how this geometric structure is reflected in stable homotopy theory.

2.3.1. Definitions. Let f be a formal group law over a ring R.

(a) We inductively define its *n-series* $[n] \in R[\![t]\!]$ of f by $[0]_f(t) = 0$ and $[n](t) = f([n-1](t), t)$ for $n \in \mathbb{Z}_{>0}$.

(b) One can show that for a prime p, if $p = 0$ in R then the p-series of f is either zero or has the form $[p](t) = rt^{p^n} + O(t^{p^n+1})$ for some nonzero $r \in R$. With a fixed prime p understood, we denote the coefficient of t^{p^n} in $[p](t)$ by v_n for $n \geq 0$. We say f has *height* $\geq n$ if $v_k = 0$ for $k < n$, and has *height (exactly) n* if furthermore v_n is invertible in R.

(c) One can show that height of a formal group law is invariant under change-of-variable, i.e. is actually a property of the underlying formal group. In fact, at each prime p we may define a stratification of $\mathscr{M}_{\mathrm{FG}} \times \mathrm{Spec}(\mathbb{Z}_{(p)})$ by defining the closed substacks $\mathscr{M}_{\mathrm{FG}}^{\geq n}$ parameterizing formal groups of height $\geq n$, whose locally closed

strata $\mathscr{M}_{\mathrm{FG}}^{n} := \mathscr{M}_{\mathrm{FG}}^{\geq n} - \mathscr{M}_{\mathrm{FG}}^{\geq n-1}$ are precisely the substacks parameterizing formal groups of height exactly n.

2.3.2. Examples. (a) It's easy to see that for any formal group law f over a ring R, the coefficient of t in $[n](t)$ is just n. In particular, f has height 0 if and only if p is invertible in R, and f has height ≥ 1 if and only if $p = 0$ in R. In other words, $\mathscr{M}_{\mathrm{FG}}^{0} \simeq \mathscr{M}_{\mathrm{FG}} \times \mathrm{Spec}(\mathbb{Q})$ and $\mathscr{M}_{\mathrm{FG}}^{\geq 1} \simeq \mathscr{M}_{\mathrm{FG}} \times \mathrm{Spec}(\mathbb{F}_p)$.

(b) Consider the additive formal group law $f(x, y) = x + y$ over a ring R with $p = 0$. Its p-series is evidently 0, so we say f has *infinite height.*

(c) Consider the multiplicative formal group law $f(x, y) = x+y+xy = (1+x)(1+y)-1$ over a ring R with $p = 0$. Its p-series is $(1 + t)^p - 1 = t^p$, so f has height 1.

Now how does the notion of height appear in stable homotopy theory? We can't give a complete answer to this question, which is central to chromatic homotopy theory, but we can say a bit.

It is often fruitful to think of abelian groups as "living over" (the spectrum of) the integers, and to study them by localizing or completing at one prime p at a time. We can do the same thing for spectra, but now there is more structure to consider than just the integer primes. We have seen that the coefficient ring of the spectrum MU is given by the Lazard ring L. Thus, for any the spectrum X, the MU-homology of X is a module $\mathrm{MU}_*(X)$ over L, i.e. a quasicoherent sheaf on Spec(L). However, more is true: the action of the group scheme G on Spec(L) described above in §2.2 lifts naturally to an action on the sheaf $\mathrm{MU}_*(X)$. Thus we may view $\mathrm{MU}_*(X)$ as a quasicoherent sheaf on the quotient stack $\mathscr{M}_{\mathrm{FG}} \simeq \mathrm{Spec}(\mathrm{L})/G$. I.e. we may think of MU-homology as a functor from the stable homotopy category to the category of quasicoherent sheaves on $\mathscr{M}_{\mathrm{FG}}$, and therefore think of the stable homotopy category as "living over" $\mathscr{M}_{\mathrm{FG}}$.

So we should ask: after localizing at a prime p, is there a way of further localizing or completing the stable homotopy category with respect to the strata of $\mathscr{M}_{\mathrm{FG}}$? Indeed there is, and describing this allows us to introduce some central objects in stable homotopy theory: the Morava K-theories and E-theories.

2.3.3. Definition. Let $\mathbb{G}_0$ be a formal group over a field κ. A *deformation* of $\mathbb{G}_0$ is a local artin ring A with residue field κ, together with a formal group $\mathbb{G}$ over A which restricts to $\mathbb{G}_0$ in the quotient map $A \to \kappa$. (Note that since κ, A are both local, the formal groups $\mathbb{G}_0, \mathbb{G}$ are just the formal groups underlying certain formal group laws f_0, f.)

2.3.4. Proposition. Let $\mathbb{G}_0$ be a formal group of height n over a perfect field κ of characteristic p. Let $\mathrm{W}(\kappa)$ denote the ring of Witt vectors of κ. Then there is a universal deformation $\mathbb{G}$ of $\mathbb{G}_0$ over the local artin ring $A := \mathrm{W}(\kappa)[\![v_1, \ldots, v_{n-1}]\!]$ with residue field κ. That is, for another local artin ring A' with residue field κ, there is a natural bijection between the set of isomorphism classes of deformations of $\mathbb{G}_0$ over A' and the set $\mathrm{Hom}_{/\kappa}(A, A')$ of ring maps $A \to A'$ over κ.

2.3.5. As noted in (2.3.3), in (2.3.4) one can think of the formal group $\mathbb{G}_0$ simply as a formal group law f_0 of height n over κ, and the universal deformation as a formal group law f over $A := \mathrm{W}(\kappa)[\![v_1, \ldots, v_{n-1}]\!]$. Essentially by construction, one can show that the universal deformation f will be Landweber-exact. Thus by (2.2.11) we have an even periodic[6] spectrum E whose associated formal group law is f, which we call the *Morava E-theory* associated to $\mathbb{G}_0$. We can say much more though: a theorem of Goerss-Hopkins-Miller[7] tells us that there is an essentially unique E_∞-ring spectrum

[6] It is automatically even periodic, not just weakly even periodic, since the base ring A is local.

[7] Disclaimer: I have not studied this theorem at all, yet.

structure on E. For the remainder, we will always think of Morava E-theory as an E_∞-ring spectrum.

On the other hand, the orginal formal group law f_0 will not be Landweber exact. Nevertheless, we may associate a spectrum to it via Morava E-theory. Namely, we let $v_0 := p \in A$, and for $0 \leq i < n$ define $M(i)$ to be the cofiber of the map of E-module spectra $E \longrightarrow E$ given by multiplication by $v_i \in A \simeq E^0$. We can then consider the tensor product $K := \bigotimes_{i=0}^{n-1} M(i)$ of E-module spectra, which is independent of the precise choice of generators $v_1, \ldots, v_n \in A$, and has coefficient ring $K^* \simeq \kappa[\beta, \beta^{-1}]$, with $\beta \in K^{-2}$. (So, just as κ is the quotient of the maximal ideal $(p, v_1, \ldots, v_{n-1}) \subseteq A$, we should think of K as the quotient of E by this maximal ideal, in some sense.) We call this spectrum K the *Morava K-theory* associated to $\mathbb{G}_0$.

We now give a definition of localizing the stable homotopy category with respect to a spectrum; localizing with respect to Morava K-theory and E-theory will realize the localization/completion to the strata of $\mathscr{M}_{\mathrm{FG}}$ we were asking for above.

2.3.6. Definition. Let E be a spectrum. We say:

- a spectrum X is *E-acyclic* if the E-homology of X vanishes, i.e. if $E_*(X) \simeq 0$ or equivalently if the smash product $E \otimes X$ is 0;
- a spectrum Y is *E-local* if any map $X \longrightarrow Y$ is nullhomotopic when X is an E-acyclic spectrum.

One can define an *E-localization functor* $L_E \colon \mathrm{Spect} \longrightarrow \mathrm{Spect}$ on the ∞-category of spectra with the following properties:

- For any the spectrum X, the spectrum $L_E X$ is E-local.
- There is a natural transformation from the identity functor on Spect to L_E, i.e. for any spectrum X there is a natural localization map $X \longrightarrow L_E X$. Moreover, this map is an isomorphism in E-homology.

2.3.7. Examples. Let $\mathbb{G}_0$ be a formal group of height n over a perfect field κ of characteristic p. Let E and K denote the associated Morava E-theory and K-theory. Then the E-localization functor L_E should be thought of as restriction to the open substack $\mathscr{M}_{\mathrm{FG}}^{\leq n}$ complementary to $\mathscr{M}_{\mathrm{FG}}^{\geq n+1}$. And the K-localization functor L_K should be thought of as completiong along the locally closed substack $\mathscr{M}_{\mathrm{FG}}^{n}$. In fact, one can show that the localization functor L_K really only depends on our chosen prime p and the height n, and for the remainder we will simply denote this functor by $L_{K(n)}$. It is also common to abusively just call these spectra *Morava E-theory and K-theory of height n*, and denote them $E(n)$ and $K(n)$, without specifying the field κ or formal group $\mathbb{G}_0$ (this is what we did in the introduction).

Here ends our discussion of the prerequisite ideas from chromatic homotopy theory that will be needed in what follows. We have certainly omitted a great deal, but hopefully the reader at this point has enough of a picture in mind that they can appreciate how our main objects of study arise.

3. p-divisible groups

The takeway of §2 is that there is a remarkable connection between stable homotopy theory and the theory of formal groups. In particular we saw that, at a fixed prime p, the stratification of formal groups by height is reflected in homotopy theory through certain spectra known at Morava K- and E-theory. It turns out that, to understand how these different heights interact in stable homotopy theory, it is extremely useful not just to consider the formal groups at hand, but to consider the p-power torsion of these formal groups as what are known as p-divisible groups.

This section reviews the bare minimum of what we'll need to know about p-divisible groups, and then analyzes the p-divisible groups in chromatic homotopy theory that we care about.

3.0.1. Notation. Throughout this section we fix a prime p.

3.1. Generalities

3.1.1. Notation. We recall some basic terminology and notation regarding (commutative) group schemes:

(a) A sequence $0 \to \mathbb{G}' \xrightarrow{i} \mathbb{G} \xrightarrow{j} \mathbb{G}'' \to 0$ of finite group schemes is called *short exact* if j is faithfully flat and i is a closed immersion which identifies $\mathbb{G}'$ with the (category-theoretic) kernel of j. If we have such a sequence such that $\mathbb{G}'$ and $\mathbb{G}''$ are finite of ranks a and b, then $\mathbb{G}$ is finite of rank ab.

(b) If we have a group scheme $\mathbb{G}$ over a ring A and a ring extension $A \to B$, we denote the base change $\mathbb{G} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ by $\mathbb{G}_B$.

(c) Similarly, if we have an ordinary group $\mathbb{G}$, we denote the associated constant group over a ring A by $\mathbb{G}_A$.

(d) Note that finite morphisms are by definition affine, so if $\mathbb{G}$ is a finite group scheme over a ring A, it is in fact an affine group scheme. In this case we denote the corresponding ring (that is, the global sections of $\mathbb{G}$) by $\mathcal{O}_{\mathbb{G}}$.

3.1.2. Definition. Let A be a commutative ring. A *p-divisible group* $\mathbb{G}$ of *height n* over A is a system $(\mathbb{G}_k, i_k)_{k \in \mathbb{Z}_{\geq 0}}$, where for each $k \in \mathbb{Z}_{\geq 0}$:

- $\mathbb{G}_k$ is a finite free[8] commutative group scheme over A of rank p^{nk};
- $i_k \colon \mathbb{G}_k \to \mathbb{G}_{k+1}$ is a morphism of group schemes over A such that the sequence

$$0 \to \mathbb{G}_k \xrightarrow{i_k} \mathbb{G}_{k+1} \xrightarrow{(p^k)} \mathbb{G}_{k+1}$$

  is exact, where (p^k) denotes multiplication by p^k; i.e. i_k identifies $\mathbb{G}_k$ as the p^k-torsion in $\mathbb{G}_{k+1}$.

It's easy to see that these morphisms in fact identify $\mathbb{G}_k$ as the p^k-torsion in $\mathbb{G}_{k+l}$ for all $l > 0$. Thus we will basically always think of $\mathbb{G}$ not as the inductive system $(\mathbb{G}_k, i_k)$, but as the colimit of this system, and denote $\mathbb{G}_k$ by $\mathbb{G}[p^k]$.

Morphisms, short exact sequences, base changes, direct sums, etc. of p-divisible groups are all defined in the obvious way in terms of their definitions for finite group schemes. Note that because rank of finite group schemes is multiplicative in short exact sequences, height of p-divisible groups is additive in short exact sequences.

3.1.3. Example. Suppose we had not a group scheme but an ordinary group $\mathbb{G}$ satisfying the axioms of a p-divisible group. Then $\mathbb{G}[p]$ would a be a finite p-torsion group of order p^n, hence would necessarily be isomorphic to $(\mathbb{Z}/p)^n$. Next, $\mathbb{G}[p^2]$ would be a finite p^2-torsion group of order p^{2n}, whose p-torsion was given by $(\mathbb{Z}/p)^n$, hence would necessarily be isomorphic to $(\mathbb{Z}/p^2)^n$. And inductively we see that the only possibility is that $\mathbb{G} \simeq \mathrm{colim}_k (\mathbb{Z}/p^k)^n \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^n$.

We deduce then that any constant p-divisible group of height n over a ring A is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^n_A$.

3.1.4. Example. Suppose we have a formal group law f over a commutative ring A. Let $\mathbb{G}_f := \mathrm{Spf}(A[\![t]\!])$ denote the associated formal group. We would like to say that

[8] Usually one says "locally free" or "flat" rather than restricting to "free" here. However, we will only deal with p-divisible groups for which these finite parts are in fact free, and we will indeed need to use this freeness hypothesis in §3.2.

the p-power torsion $\mathbb{G}_f[p^\infty] := \operatorname{colim}_k \mathbb{G}_f[p^k]$ of $\mathbb{G}_f$ is a p-divisible group, where $\mathbb{G}_f[p^k] \simeq \operatorname{Spec}(A[\![t]\!]/([p^k](t)))$ with $[p^k](t) \in A[\![t]\!]$ the p^k-series of f. A form of the Weierstrass preparation theorem [5, 5.1–5.2] tells us that if A is complete with respect to an ideal I, and the p-series of f satisfies $[p](t) \equiv ut^{p^n} \mod (I, t^{p^n+1})$ for a unit $u \in A^\times$ and $n \geq 1$, then the A-algebra $A[\![t]\!]/([p^k](t))$ is a free A-module with basis $\{1, t, \ldots, t^{p^{nk}-1}\}$ for all $k \geq 0$. So under these hypotheses, the p-power torsion $\mathbb{G}_f[p^\infty]$ will indeed be a p-divisible group, of height n.

Often it is possible to decompose an arbitrary p-divisible group $\mathbb{G}$ into ones which look like the two examples above. Namely, one often has a short exact sequence

$$0 \to \mathbb{G}_{\mathrm{inf}} \to \mathbb{G} \to \mathbb{G}_{\text{ét}} \to 0, \tag{3.1.5}$$

where $\mathbb{G}_{\mathrm{inf}}$ is an "infinitesimal" p-divisible group arising from some formal group law, and $\mathbb{G}_{\text{ét}}$ is étale (hence fairly close to being constant). In §3.2 we will analyze the behavior of such short exact sequences, specifically how we can extend our base ring such that $\mathbb{G}_{\text{ét}}$ actually becomes constant and such that the sequence splits. The following elementary observation will be useful in this analysis.

3.1.6. Proposition. Let $\mathbb{G}$ be a p-divisible group over a ring A. Suppose we have a short exact sequence (3.1.5). Let r be the height of $\mathbb{G}_{\text{ét}}$. The following are equivalent:

(a) the exact sequence splits and the étale part of $\mathbb{G}$ is constant, i.e. $\mathbb{G} \simeq \mathbb{G}_{\mathrm{inf}} \oplus \mathbb{G}_{\text{ét}}$ and $\mathbb{G}_{\text{ét}} \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^r_A$;

(b) there is a map $(\mathbb{Q}_p/\mathbb{Z}_p)^r_A \to \mathbb{G}$ such that the composite $(\mathbb{Q}_p/\mathbb{Z}_p)^r_A \to \mathbb{G}_{\text{ét}}$ is an isomorphism.

Proof. Clearly (a) implies (b). Conversely, assuming (b), we get a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{G}_{\mathrm{inf}} \arrow[r] \arrow[d] & \mathbb{G}_{\mathrm{inf}} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^r_A \arrow[r] \arrow[d] & (\mathbb{Q}_p/\mathbb{Z}_p)^r_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{G}_{\mathrm{inf}} \arrow[r] & \mathbb{G} \arrow[r] & \mathbb{G}_{\text{ét}} \arrow[r] & 0.
\end{tikzcd}$$

The left vertical map is just the identity, and the right vertical map is by hypothesis an isomorphism. We conclude by applying the 5-lemma. □

3.2. Splitting the connected-étale sequence

In this subsection we construct the universal extension of a ring over which a given p-divisible group splits as the direct sum of its infinitesimal part and a constant étale part. We essentially follow Stapleton [13, §2.8], who essentially follows Hopkins-Kuhn-Ravenel [5, §§6.1–6.2]. However, rather than restrict to the specific example which will be relevant to character theory, here we isolate this part of the argument which applies more generally; I think this generality actually clarifies the exposition slightly.

3.2.1. Notation. Throughout this subsection we let $\mathbb{G}$ be a p-divisible group of height n over a ring A. Assume that we have a short exact sequence of p-divisible groups

$$0 \to \mathbb{G}_{\mathrm{inf}} \to \mathbb{G} \to \mathbb{G}_{\text{ét}} \to 0$$

where $\mathbb{G}_{\mathrm{inf}}$ has height t and $\mathbb{G}_{\text{ét}}$ is étale of height $n - t$.

Let $\Lambda := (\mathbb{Q}_p/\mathbb{Z}_p)^{n-t}$.[9] For all $k \geq 0$ we may choose generators $\lambda_1^k, \ldots, \lambda_{n-t}^k$ of $\Lambda[p^k] \simeq (\mathbb{Z}/p^k)^{n-t}$ which are coherent in the sense that $\lambda_i^k = p\lambda_i^{k+1}$ for $1 \leq i \leq n - t$.

[9] Warning: our Λ is denoted $\Lambda^\vee$ in [5, 13], and thus our $\Lambda^\vee$ (appearing in §3.3) is their Λ.

3.2.2. Lemma. For $k \geq 0$, the functor $\mathrm{Alg}_A \to \mathrm{Set}$ assigning to an A-algebra B the set $\mathrm{Hom}_B(\Lambda[p^k]_B, \mathbb{G}[p^k]_B)$ of morphisms of group schemes $\Lambda[p^k]_B \to \mathbb{G}[p^k]_B$ over B is corepresented by the A-algebra

$$C_k := (\mathcal{O}_{\mathbb{G}[p^k]})^{\otimes(n-t)} := \mathcal{O}_{\mathbb{G}[p^k]} \otimes_A \cdots \otimes_A \mathcal{O}_{\mathbb{G}[p^k]};$$

i.e. any morphism $\Lambda[p^k]_B \to \mathbb{G}[p^k]_B$ is the base change of a universal morphism $\phi_{\mathrm{univ}} \colon \Lambda[p^k]_{C_k} \to \mathbb{G}[p^k]_{C_k}$ in a unique map $C_k \to B$.

The natural transformation

$$\mathrm{Hom}_B(\Lambda[p^{k+1}]_B, \mathbb{G}[p^{k+1}]_B) \to \mathrm{Hom}_B(\Lambda[p^k]_B, \mathbb{G}[p^k]_B)$$

given by restriction corresponds via the Yoneda lemma to the morphism $C_k \to C_{k+1}$ induced by the multiplication-by-p map $\mathbb{G}[p^{k+1}] \to \mathbb{G}[p^k]$.

Proof. Let $B \in \mathrm{Alg}_A$. Our chosen generators $\lambda_1^k, \ldots, \lambda_{n-t}^k$ of $\Lambda[p^k]$ determine an isomorphism $\Lambda[p^k] \simeq (\mathbb{Z}/p^k)^{n-t}$. Therefore restricting to the factors in the constant group scheme

$$\Lambda[p^k]_B \simeq \coprod_{\lambda \in \Lambda[p^k]} \mathrm{Spec}(B)$$

indexed by these generators determines a bijection between morphisms of group schemes $\Lambda[p^k]_B \to \mathbb{G}[p^k]_B$ over B and $(n-t)$-tuples of morphisms of schemes $\mathrm{Spec}(B) \to \mathbb{G}[p^k]_B$ over B. The latter are equivalent to morphisms $\mathrm{Spec}(B) \to \mathbb{G}[p^k]$ over A. Since $\mathbb{G}[p^k]$ is affine, this equivalent to specifying a morphism of A-algebras $(\mathcal{O}_{\mathbb{G}[p^k]})^{\otimes(n-t)} \to B$. We have now described a bijection $\mathrm{Hom}_B(\Lambda[p^k]_B, \mathbb{G}[p^k]_B) \simeq \mathrm{Hom}_{\mathrm{Alg}_A}(C_k, B)$ which clearly is natural in B, proving the claim. Note that the universal morphism $\phi_{\mathrm{univ}} \colon \Lambda[p^k]_{C_k} \to \mathbb{G}[p^k]_{C_k}$ corresponds to the identity $\mathrm{id} \colon C_k \to C_k$ under this bijection.

The final statement about the induced map $C_k \to C_{k+1}$ is immediate from the coherence of the generators $\lambda_1^k, \ldots, \lambda_{n-t}^k$ fixed in (3.2.1). □

3.2.3. Lemma. For $k \geq 0$, the functor $\mathrm{Alg}_A \to \mathrm{Set}$ assigning to an A-algebra B the subset $\mathrm{Iso}_B(\Lambda[p^k]_B, \mathbb{G}_{\text{ét}}[p^k]_B) \subseteq \mathrm{Hom}_B(\Lambda[p^k]_B, \mathbb{G}[p^k]_B)$ consisting of morphisms $\Lambda[p^k]_B \to \mathbb{G}[p^k]_B$ for which the composite morphism

$$\Lambda[p^k]_B \to \mathbb{G}[p^k]_B \to \mathbb{G}_{\text{ét}}[p^k]_B$$

is an isomorphism is corepresented by the localization $C_k[\Delta_k^{-1}]$, for some element $\Delta_k \in C_k$. More precisely, we have a commutative diagram of natural transformations

$$\begin{tikzcd} \mathrm{Iso}_B(\Lambda[p^k]_B, \mathbb{G}_{\text{ét}}[p^k]_B) \arrow[r, hook] \arrow[d, "\wr"] & \mathrm{Hom}_B(\Lambda[p^k]_B, \mathbb{G}[p^k]_B) \arrow[d, "\wr"] \\ \mathrm{Hom}_A(C_k[\Delta_{t,k}^{-1}], B) \arrow[r, hook] & \mathrm{Hom}_A(C_k, B), \end{tikzcd} \tag{3.2.4}$$

where the bottom map is the canonical inclusion and the right map is the isomorphism found in (3.2.2). Finally, $C_k[\Delta_{t,k}^{-1}]$ is faithfully flat over A.

Proof. Suppose we have a morphism $\phi \colon \Lambda[p^k]_B \to \mathbb{G}[p^k]_B$. By (3.2.2) this determines a unique map $\alpha \colon C_k \to B$ in which ϕ is the base change of ϕ_{univ}. Consider the composite

$$\Lambda[p^k]_{C_k} \xrightarrow{\phi_{\mathrm{univ}}} \mathbb{G}[p^k]_{C_k} \to \mathbb{G}_{\text{ét}}[p^k]_{C_k}, \tag{3.2.5}$$

which corresponds to a morphism of C_k-algebras

$$\mathcal{O}_{\mathbb{G}_{\text{ét}}[p^k]_{C_k}} \to \mathcal{O}_{\Lambda[p^k]_{C_k}}. \tag{3.2.6}$$

By hypothesis and definition, both algebras are finite free of the same rank, so we may consider its determinant $\Delta_k \in C_k$. Now, everything in sight is affine, so the base change $\Lambda[p^k]_B \to \mathbb{G}_{\text{ét}}[p^k]_B$ of (3.2.5) in α, which by definition of α is the composite

$$\Lambda[p^k]_B \xrightarrow{\phi} \mathbb{G}[p^k]_B \to \mathbb{G}_{\text{ét}}[p^k]_B,$$

is an isomorphism if and only if the base change $\mathcal{O}_{\mathbb{G}_{\text{ét}}[p^k]_B} \to \mathcal{O}_{\Lambda[p^k]_B}$ of (3.2.6) in α is an isomorphism. But of course this is true if and only if $\alpha(\Delta_k)$ is a unit in B, i.e. if α factors through the localization $C_k[\Delta_k^{-1}]$. This proves the existence and commutativity of the diagram (3.2.4).

Finally we prove faithful flatness. Since $\mathcal{O}_{\mathbb{G}[p^k]}$ is finite free over A, so is C_k. And localization is flat so this implies $C_k[\Delta_k^{-1}]$ is flat over A. So to prove faithful flatness we just need to show that $\operatorname{Spec}(C_k[\Delta_k^{-1}]) \to \operatorname{Spec}(A)$ is surjective. Let $\mathfrak{p} \in \operatorname{Spec}(A)$; let K be the algebraic closure of the fraction field of the domain $A/\mathfrak{p}$, so we have a map $\beta\colon A \to K$ with kernel $\mathfrak{p}$. Since $\mathbb{G}_{\text{ét}}$ is étale, it must be constant when base-changed to K. So by (3.1.3) there must be an isomorphism $\Lambda[p^k]_K \xrightarrow{\sim} \mathbb{G}_{\text{ét}}[p^k]_K$. Since K is algebraically closed, there is necessarily a map $\Lambda[p^k]_K \to \mathbb{G}[p^k]_K$ lifting this isomorphism.[10] So by the above, β must factor through a map $\gamma\colon C_k[\Delta_k^{-1}] \to K$. If we set $\mathfrak{q} := \ker(\gamma) \in \operatorname{Spec}(C_k[\Delta_k^{-1}])$ then $\mathfrak{q}$ restricts to $\mathfrak{p}$ in $\operatorname{Spec}(A)$. Since $\mathfrak{p}$ was arbitrary, we have the desired surjectivity. □

3.2.7. Remark. By the commutativity of (3.2.4), the maps $C_k \to C_{k+1}$ defined in (3.2.2) also induce maps $C_k[\Delta_k^{-1}] \to C_{k+1}[\Delta_{k+1}^{-1}]$ on the localizations defined in (3.2.3).

3.2.8. Proposition. The functor $\mathrm{Alg}_A \to \mathrm{Set}$ assigning to an A-algebra B the subset $\operatorname{Iso}_B(\Lambda_B, (\mathbb{G}_{\text{ét}})_B) \subseteq \operatorname{Hom}_B(\Lambda_B, \mathbb{G}_B)$ consisting of morphisms $\Lambda_B \to \mathbb{G}_B$ for which the composite morphism

$$\Lambda_B \to \mathbb{G}_B \to (\mathbb{G}_{\text{ét}})_B$$

is an isomorphism is corepresented by $C := \operatorname{colim}_k C_k[\Delta_k^{-1}]$, where C_k and Δ_k are as in (3.2.2, 3.2.3).

Proof. By definition of a morphism of p-divisible groups,

$$\begin{aligned}\operatorname{Iso}_B(\Lambda_B, (\mathbb{G}_{\text{ét}})_B) &\simeq \lim_k \operatorname{Iso}_B(\Lambda[p^k]_B, \mathbb{G}_{\text{ét}}[p^k]_B)\\ &\simeq \lim_k \operatorname{Hom}_A(C_k[\Delta_k^{-1}], B)\\ &\simeq \operatorname{Hom}_A(\operatorname{colim}_k C_k[\Delta_k^{-1}], B),\end{aligned}$$

as desired. □

3.3. Chromatic examples

We now discuss the p-divisible groups associated to Morava E-theory and its $K(t)$-localization. Some proofs and computations in this section are omitted/cited.[11]

3.3.1. Let E be the Morava E-theory associated to a formal group $\mathbb{G}_0$ of height n over a perfect field κ of characteristic p. As discussed in (2.3.5), E is even periodic with coefficient ring given by

$$E^0 \simeq \mathrm{W}(\kappa)[\![v_1, \dots, v_{n-1}]\!].$$

[10] This is a fact which I read in [1, p. 32], and which is proved in [2, III, 3.7.6].

[11] These omissions are unfortunate, but this thesis has a due date, which is just as unfortunate.

As shown in [10], the formal group law (over E^0) associated to E can be taken such that its p-series satisfies the following: for $0 \le t < n$,

$$[p](x) \equiv v_t x^{p^t} \mod (I_t, x^{p^t+1}), \tag{3.3.2}$$

where I_t denotes the ideal $(p, v_1, \dots, v_{t-1})$. For $t = n$, the assumption that E is of height n implies that

$$[p](x) \equiv v_n x^{p^n} \mod (I_n, t^{p^n+1}), \tag{3.3.3}$$

where $v_n \in E^0$ is a unit, and now I_n is the maximal ideal $(p, v_1, \dots, v_{n-1})$ in the local ring E^0. Since E^0 is complete with respect to I_n, (3.1.4) implies that the p-power torsion of the formal group of E (the universal deformation of $\mathbb{G}_0$) is a p-divisible group of height n. We will denote this p-divisible group by $\mathbb{G}_E$.

3.3.4. Now, fix $0 \le t < n$. We consider the spectrum $L_t := L_{K(t)}E(n)$, which has the structure of an E_∞-ring because E does. One can show that L_t is also an even periodic spectrum whose coefficient ring is obtained from E^0 by inverting v_t and then completing with respect to the ideal I_t; that is,

$$L_t^0 \simeq \mathrm{W}(\kappa)[\![v_1, \dots, v_{n-1}]\!][v_t^{-1}]_{I_t},$$

and the localization map $E \to L_t$ induces the canonical map of coefficient rings $E^0 \to L_t^0$. We let I_t denote the ideal $(p, v_1, \dots, v_{t-1})$ in L_t^0 as well (which should not be too confusing). The formal group law associated to the spectrum L_t is obtained simply by applying the canonical map $E^0 \to L_t^0$ to the coefficients of the formal group law associated to E. In particular, the congruence (3.3.2) still holds for the formal group law of L_t. Since v_t is invertible in L_t^0 and L_t^0 is complete with respect to I_t, (3.1.4) implies that the p-power torsion of the formal group of L_t is a p-divisible group of height t. We will denote this p-divisible group by $\mathbb{G}_{L_t}$.

3.3.5. Remark. Maybe it seems abusive to denote the p-divisible groups above by $\mathbb{G}_E$ and $\mathbb{G}_{L_t}$, since it looks like they denote the entire formal group. One reason this isn't so bad is that *all* the torsion in these formal groups is necessarily p-power torsion. As mentioned earlier, for $s > 0$ the s-series of a formal group law is always of the form $[s](t) = st + O(t^2)$. If s is coprime to p, then it is invertible in the rings E^0 and L_t^0, so $[s]$ is an invertible power series, and hence the formal group can have no s-torsion.

The next result states that torsion of the formal groups associated to E and L_t, i.e. the p-divisible groups $\mathbb{G}_E$ and $\mathbb{G}_{L_t}$, can be expressed purely in terms of cohomology.

3.3.6. Proposition ([5, §5.4]). Let F denote either the spectrum E or L_t.

(a) Let $m > 0$. Then

$$F^*(\mathrm{B}\mathbb{Z}/m) \simeq F^*[\![t]\!]/([m](t)),$$

where $[m]$ denotes the m-series of the formal group law associated to F. In particular, for $m = p^k$ we have $F^0(\mathrm{B}\mathbb{Z}/p^k) \simeq \mathcal{O}_{\mathbb{G}_F[p^k]}$, i.e. $\mathrm{Spec}(F^0(\mathrm{B}\mathbb{Z}/p^k)) \simeq \mathbb{G}_F[p^k]$.

(b) Moreover, if $m = sp^k$ with s coprime to p, then the map

$$F^*[\![t]\!]/([m](t)) \simeq F^*(\mathrm{B}\mathbb{Z}/m) \to F^*(\mathrm{B}\mathbb{Z}/p^k) \simeq F^*[\![t]\!]/([p^k](t))$$

is an isomorphism.

(c) Most generally, if A is a finite abelian group whose subgroup of p-torsion is given by $A[p^\infty] \simeq \bigoplus_i \mathbb{Z}/p^{k_i}$, then

$$F^*(\mathrm{B}A) \simeq F^*(\mathrm{B}A[p^\infty]) \simeq \bigotimes_i F^*(\mathrm{B}\mathbb{Z}/p^{k_i}),$$

implying $F^0(\mathrm{B}A) \simeq \bigotimes_i \mathscr{O}_{\mathbb{G}_F[p^{k_i}]}$.

Idea of proof. What (a) says intuitively is that the fact that $\mathbb{Z}/m$ is the m-torsion in S^1 is still visible after applying F-cohomology, most clearly when the cohomology rings are viewed algebro-geometrically. More precisely, the exact sequence $\mathbb{Z}/m \to \mathrm{S}^1 \to \mathrm{S}^1$ gives us a fiber sequence $\mathrm{B}\mathbb{Z}/m \to \mathrm{BS}^1 \to \mathrm{BS}^1$, and the above result tells us that this sequence identifies $\mathrm{Spec}(F^*(\mathrm{B}\mathbb{Z}/m))$ with the m-torsion in the formal group $\mathrm{Spf}(F^*(\mathrm{BS}^1)) \simeq \mathrm{Spf}(F^*(\mathbb{CP}^\infty)) \simeq \mathrm{Spf}(F^*[\![t]\!])$ associated to F.

Then (b) just follows from the fact that the formal groups we are considering only have p-power torsion, as discussed in (3.3.5). Finally, (c) follows from a Kunneth isomorphism, which exists because we know the rings $\mathscr{O}_{\mathbb{G}_F[p^k]}$ are finite free over F^0. □

3.3.7. The map of ring spectra $E \to L_t$ gives natural maps in cohomology $E^0(\mathbb{B}\mathbb{Z}/p^k) \to L_t(\mathbb{B}\mathbb{Z}/p^k)$ for each k, which induce maps

$$L_t \otimes_{E^0} E^0(\mathbb{B}\mathbb{Z}/p^k) \to L_t(\mathbb{B}\mathbb{Z}/p^k).$$

Thus, if we let $\mathbb{G} := (\mathbb{G}_E)_{L_t^0}$ be the base change of $\mathbb{G}_E$ along $E^0 \to L_t^0$, (3.3.6) implies that these maps determine a map of p-divisible groups $i\colon \mathbb{G}_{L_t} \to \mathbb{G}$ over L_t^0.

3.3.8. **Proposition ([13, §2.1]).** The map $i\colon \mathbb{G}_{L_t} \to \mathbb{G}$ is injective with étale quotient. That is, we have an exact sequence

$$0 \to \mathbb{G}_{L_t} \xrightarrow{i} \mathbb{G} \to \mathbb{G}_{\text{ét}} \to 0$$

of p-divisible groups over L_t^0, with $\mathbb{G}_{\text{ét}}$ étale of height $n-t$.

3.3.9. We can now apply the general analysis carried out in §3.2 to the exact sequence given by (3.3.8). Let $C^0_{t,k} := (\mathscr{O}_{\mathbb{G}[p^k]})^{\otimes(n-t)}$, and $\Delta_{t,k} \in C^0_{t,k}$ the determinant inverted in (3.2.3). Then (3.2.8) tells us that $\mathbb{G}$ splits into a direct sum of $\mathbb{G}_{L_t}$ and the constant group $\Lambda := (\mathbb{Q}_p/\mathbb{Z}_p)^{n-t}$ over $C^0_t := \mathrm{colim}_k\, C^0_{t,k}[\Delta_{t,k}^{-1}]$. More precisely, there is a canonical isomorphism of p-divisible groups

$$(\mathbb{G}_{L_t})_{C^0_t} \oplus \Lambda_{C^0_t} \xrightarrow{\sim} \mathbb{G}_{C^0_t}. \tag{3.3.10}$$

Algebraically this is an isomorphism of Hopf algebras

$$C^0_t \otimes_{L^0_t} \mathscr{O}_{\mathbb{G}[p^k]} \xrightarrow{\sim} \left(C^0_t \otimes_{L^0_t} \mathscr{O}_{\mathbb{G}_{L_t}[p^k]}) \otimes_{C^0_t} \mathscr{O}_{\Lambda[p^k]_{C^0_t}}\right)$$

for all k. Let's unwrap a few definitions:

- $\mathscr{O}_{\mathbb{G}[p^k]} \simeq L^0_t \otimes_{E^0} \mathscr{O}_{\mathbb{G}_E[p^k]} \simeq L^0_t \otimes_{E^0} E^0(\mathrm{B}\mathbb{Z}/p^k)$, the second isomorphism coming from (3.3.6);
- $\mathscr{O}_{\mathbb{G}_{L_t}[p^k]} \simeq L^0_t(\mathrm{B}\mathbb{Z}/p^k)$, agian by (3.3.6);
- $\mathscr{O}_{\Lambda[p^k]_{C^0_t}} \simeq \bigoplus_{\lambda\in\Lambda[p^k]} C^0_t$.

So we may rewrite the above isomorphism of Hopf algebras as

$$C^0_t \otimes_{E^0} E^0(\mathrm{B}\mathbb{Z}/p^k) \xrightarrow{\sim} \left(C^0_t \otimes_{L^0_t} L^0_t(\mathrm{B}\mathbb{Z}/p^k)\right) \otimes_{C^0_t} \bigoplus_{\lambda\in\Lambda[p^k]} C^0_t \tag{3.3.11}$$

$$\simeq C^0_t \otimes_{L^0_t} \bigoplus_{\lambda\in\Lambda[p^k]} L^0_t(\mathrm{B}\mathbb{Z}/p^k). \tag{3.3.12}$$

Note finally that by (3.3.6), there is an isomorphism $C^0_{t,k} \simeq L^0_t(\mathrm{B}\Lambda^\vee_k)$, where $\Lambda^\vee_k := (\mathbb{Z}/p^k)^{n-t}$. The maps $C^0_{t,k} \to C^0_{t,k+1}$ in the colimit defining C_t are then just induced by the canonical maps $\Lambda^\vee_{k+1} \to \Lambda^\vee_k$.

Let's now understand the above isomorphisms in terms of cohomology. We have two factors:

- The first is given by the map $(\mathbb{G}_{L_t})_{C_t^0} \to \mathbb{G}_{C_t^0}$ in (3.3.10), which algebraically is given by the map $C_t^0 \otimes_{E^0} E^0(\mathrm{B}\mathbb{Z}/p^k) \to C_t^0 \otimes_{L_t^0} L_t^0(\mathrm{B}\mathbb{Z}/p^k)$ in (3.3.11). This is just the base change of our inclusion $\mathbb{G}_{L_t} \hookrightarrow \mathbb{G}$, which by definition is algebraically just induced by the canonical map $E^0(\mathrm{B}\mathbb{Z}/p^k) \to L_t^0(\mathrm{B}\mathbb{Z}/p^k)$.
- The second is given by the map $\Lambda_{C_t^0} \to \mathbb{G}_{C_t^0}$ in (3.3.10), which algebraically is given by the map $C_t^0 \otimes_{E^0} E^0(\mathrm{B}\mathbb{Z}/p^k) \to \bigoplus_{\lambda \in \Lambda[p^k]} C_t^0$ in (3.3.11). This is determined by maps $\phi_\lambda \colon E^0(\mathrm{B}\mathbb{Z}/p^k) \to C_t^0$ for each $\lambda \in \Lambda[p^k]$. The definition of ϕ_λ comes from the definition of C_t^0. We had a sequence of fixed generators $\lambda_1^k, \dots, \lambda_{n-t}^k \in \Lambda[p^k]$. So first suppose $\lambda = \lambda_i^k$. Then ϕ_λ is the composition

$$E^0(\mathrm{B}\mathbb{Z}/p^k) \to L_t^0(\mathrm{B}\mathbb{Z}/p^k) \xrightarrow{\psi_\lambda} L_t^0(\mathrm{B}\mathbb{Z}/p^k)^{\otimes(n-t)} \simeq C_{t,k} \to C_{t,k}[\Delta_{t,k}^{-1}] \to C_t^0$$

  where ψ_λ is the inclusion of the i-th factor of the tensor product. More generally, if $\lambda = \sum_i a_i \lambda_i^k$ then ψ_λ will be the corresponding linear combination of inclusions $\sum_i a_i \psi_{\lambda_i^k}$, where now summation refers to the formal group operation on $L_t^0(\mathrm{B}\mathbb{Z}/p^k) \simeq \mathcal{O}_{\mathbb{G}_{L_t}[p^k]}$. We can rephrase this using our identification

$$C_{t,k}^0 \simeq L_t^0(\mathrm{B}\mathbb{Z}/p^k)^{\otimes(n-t)} \simeq L_t^0(\mathrm{B}\Lambda_k^\vee).$$

  Then ψ_λ is simply induced by the homomorphism $\Lambda_k^\vee \to \mathbb{Z}/p^k$ which sends $(x_1, \dots, x_{n-t}) \mapsto \sum a_i x_i$.

Ok, that was was a fairly long block of algebra and isomorphism-chasing, but we can cleanly record (a slight generalization of) our conclusion as follows.

3.3.13. Proposition. Let A be a finite abelian group, and let $A[p^\infty] \subseteq A$ denote the subgroup of p-torsion. Since A is finite, we can in fact pick a finite k for which $A[p^\infty] = A[p^k]$. Then to each tuple $a = (a_1, \dots, a_{n-t}) \in A[p^\infty]^{n-t}$ we can associate the homomorphism $\Lambda_k^\vee \simeq (\mathbb{Z}/p^k)^{n-t} \to A$ which sends $(x_1, \dots, x_{n-1}) \mapsto \sum_i a_i x_i$. This determines a map

$$E^0(\mathrm{B}A) \to E^0(\mathrm{B}\Lambda_k^\vee) \to L_t^0(\mathrm{B}\Lambda_k^\vee) \simeq C_{t,k}^0 \to C_t^0,$$

which is independent of our choice of k. These maps, along with the canonical map $E^0(\mathrm{B}A) \to L_t^0(\mathrm{B}A)$, determine an isomorphism

$$\begin{aligned} C_t^0 \otimes_{E^0} E^0(\mathrm{B}A) &\xrightarrow{\sim} \left(C_t^0 \otimes_{L_t^0} L_t^0(\mathrm{B}A)\right) \otimes_{C_t^0} \bigoplus_{a \in A[p^\infty]^{n-t}} C_t^0 \\ &\simeq C_t^0 \otimes_{L_t^0} \bigoplus_{a \in A[p^\infty]^{n-t}} L_t^0(\mathrm{B}A), \end{aligned} \tag{3.3.14}$$

natural in A.

Proof. The naturality of the defined map is clear. To prove it is an isomorphism we can pick an isomorphism $A[p^\infty] \simeq \bigoplus \mathbb{Z}/p^{k_i}$, and using (3.3.6) reduce to the case $A = \mathbb{Z}/p^k$. Then the claim is precisely what was shown above in (3.3.9). □

We now show that we can lift this result from cohomology to an equivalence at the level of E_∞-ring spectra.

3.3.15. Construction. For each k we may define an E_∞-ring

$$C_{t,k} \coloneqq L_t(\mathrm{B}\Lambda_k^\vee),$$

the function spectrum whose homotopy groups give the cohomology ring $L_t^*(\mathrm{B}\Lambda_k^\vee)$. In particular, its π_0 agrees with what we have already been calling $C_{t,k}^0$. It is then possible to construct an E_∞-ring $C_{t,k}[\Delta_{t,k}^{-1}]$ whose π_0 is precisely $C_{t,k}^0[\Delta_{t,k}^{-1}]$. Taking the colimit over the canonical maps $\mathrm{B}\Lambda_{k+1}^\vee \to \mathrm{B}\Lambda_k^\vee$, we may define

$$C_t := \operatorname*{colim}_k C_{t,k}[\Delta_{t,k}^{-1}]$$

to obtain an E_∞-ring whose π_0 is precisely C_t^0 from above.

3.3.16. Lemma. The E_∞-ring C_t is even periodic, and the E_∞-ring maps $E \to L_t \to C_t$ are flat.

Proof. To see that C_t is even periodic it suffices to show that each $C_{t,k}$ is even periodic, which is immediate from the computation (3.3.6) of $L_t^*(\mathrm{B}\Lambda_k^\vee)$. Since E and L_t are also even periodic, it suffices to check flatness in π_0. But this we know already: L_t^0 is obtained by from E^0 by localization and then completion (of a noetherian ring), hence flat; and C_t^0 is flat over L_t^0 since each $C_{t,k}^0$ is (faithfully) flat over L_t^0, as proved in (3.2.3). □

3.3.17. Proposition. The isomorphism (3.3.14) can be lifted from π_0 to an equivalence of E_∞-rings

$$C_t \otimes_E E(\mathrm{B}A) \simeq C_t \otimes_{L_t} \bigoplus_{a \in A[p^\infty]} L_t(\mathrm{B}A),$$

natural in A.

Proof. First note that π_0 of the left- and right-hand sides actually recover the left- and right-hand sides of (3.3.11) because everything in sight is flat. Now, the isomorphism (3.3.14) was determined by maps in cohomology, induced by maps between classifying spaces. Since the localization map $E \to L_t$ is a map of E_∞-rings, it is evident that these maps can be lifted to E_∞-maps on the associated function spaces. So we automatically get the desired map of E_∞-rings, which by construction is an isomorphism on π_0. But by (3.3.6, 3.3.16) both sides are even periodic, so it must then be an equivalence. □

4. Global equivariant homotopy theory

Back in §1 we translated the character theory of a finite group G into a statement in G-equivariant homotopy theory. If we want to take this perspective seriously as a strategy for giving a *proof* of character theory—well, more interestingly, for giving proofs of generalizations of character theory—it will turn out to be crucial that we can make this statement for *all* finite groups G, and that there is moreover some relationship among the statements for all these groups. This idea should be familiar from the classical setting of representation theory: one often tries to get a handle on the representations of a group G by considering the representations induced from subgroups $H \subseteq G$ (see for example the proof of (1.1.5)!).

To systematically handle such phenomena, we want some kind of homotopy theory in which we can study equivariance for all finite[12] groups simultaneously. This idea goes by the name of *global equivariant homotopy theory*. This section is devoted to setting up the basic framework of global equivariant homotopy theory.[13]

[12] We restrict ourselves to finite groups because these are the groups which will be relevant to our discussion later on. In other situations one might be interested in, say, all compact Lie groups. The framework we set up in this section should carry over without trouble in such situations.

[13] The first time my eyes saw the global perspective was when reading Lurie's survey [6], but the ideas are left ever so slightly implicit there and thus my brain didn't process what was really happening for

4.1. The global indexing category

To guide our definition of global equivariant homotopy theory, we recall the setup of (unstable) G-equivariant homotopy theory for a fixed group G. This is summarized by the following theorem, often referred to as Elmendorf's theorem. We currently don't have any time to discuss its proof, but one can see [12] for more, including a detailed list of references.

4.1.1. Notation. For G a finite group, let $\mathrm{Orb}(G)$ denote *the category of G-orbits*, whose objects are transitive G-sets and morphisms are G-equivariant maps of such sets. Any object $T \in \mathrm{Orb}(G)$ is of course isomorphic to a G-set of the form G/H for some subgroup $H \subseteq G$, but choosing such an isomorphism amounts to choosing a basepoint in T. We will often be slightly evil/informal and refer to a orbit directly as G/H, i.e. think of the objects of $\mathrm{Orb}(G)$ as pointed, since this is often more intuitive.

4.1.2. Theorem (Elmendorf). Let G be a finite group. The following homotopy theories[14] are all equivalent:

(a) G-Top: the relative category of topological spaces equipped with a G-action, with weak equivalences defined to be the (G-equivariant) maps $X \longrightarrow Y$ which induce weak equivalences (of topological spaces) on the fixed-point spaces $X^H \longrightarrow Y^H$ for all subgroups $H \subseteq G$. (In fact there is a standard model structure on this relative category.)

(b) G-CW: the relative category of G-CW-complexes, with weak equivalences defined to be the (G-equivariant) homotopy equivalences. (In fact this is just the subcategory of fibrant-cofibrant objects in the model structure on G-Top mentioned above.)

(c) $\mathrm{Space}_{\mathrm{Orb}(G)}$: the homotopy theory of presheaves of spaces on the category of G-orbits $\mathrm{Orb}(G)$, i.e. the ∞-category of functors $\mathrm{Orb}(G)^{\mathrm{op}} \longrightarrow \mathrm{Space}$.

The equivalence of (a) and (c) is exhibited by the functor $G\text{-Top} \longrightarrow \mathrm{Space}_{\mathrm{Orb}(G)}$ sending $X \in G\text{-Top}$ to the presheaf $T \longmapsto \mathrm{Map}_{G\text{-Top}}(T, X)$, or in other words $G/H \longmapsto X^H$.

4.1.3. Notation. We'll denote the equivalent homotopy theories described in (4.1.2) by G-Space when we want to make a statement without evoking any particular model.

As far as I understand, it is formulation (4.1.2)(b) of G-Space which makes it interesting and useful: it turns out that G-CW-complexes up to homotopy equivalence effectively model the spaces of interest to us, e.g. smooth G-manifolds (see [14] for some discussion about this, and (5.2.8) for an example of us critically using this fact). However, the presheaf formulation (4.1.2)(c) is quite convenient from a categorical perspective: $\mathrm{Space}_{\mathrm{Orb}(G)}$ comes to us immediately as an ∞-category, in terms of the ∞-category of spaces, allowing us to speak very easily about limits and colimits for example. Such a framework will also be convenient in the global setting (evidence abounds in this section and the next), inspiring us to formulate global equivariant homotopy theory also as the homotopy theory of presheaves of spaces on some indexing category Glo. Before giving the definition of Glo, we give some intuition guiding what the definition should be.

4.1.4. Intuition. (a) Unstably, we want to somehow assemble the homotopy theories G-Space for all G. So a natural starting point is to ask: how are all these homotopy

quite some time. What shed light on this picture for me was a note of Rezk [11], from which I have learned much of what is included in this section (and the presentation here pretty clearly reflects that).

[14] We use the term "homotopy theory" synonymously with ∞-category, except more informally. For example, relative categories and model categories present ∞-categories, but aren't yet ∞-categories. Yet we refer to all of these things as "homotopy theories".

theories related? Of course, whenever we have a group homomorphism $\phi\colon H \to G$, we can pull back an action of G on a space to an action of H, giving us a restriction functor $\phi^*\colon G\text{-Space} \to H\text{-Space}$. If this were all there were to the answer, then we could just define Glo to be the category of finite groups. However there is something else to remember: whenever $\psi\colon H \to G$ is a G-conjugate $g\phi g^{-1}$ of $\phi\colon H \to G$, there is a natural equivalence $\phi^* \xrightarrow{\sim} \psi^*$, given on a G-space X by the action $g\colon \phi^*(X) \to \psi^*(X)$.

We can summarize the above discussion as follows. If we view our groups G as groupoids in the usual manner, then Space_G is functorial not only in category of groups, but in the 2-*category* of such groupoids (where the 1-morphisms are functors, corresponding to group homomorphisms, and the 2-morphisms are natural transformations, corresponding to conjugation of group homomorphisms). We will spell this out in detail in (4.1.5, 4.1.6) to give our definition of Glo.

(b) Stably, intuition comes from the most basic example of global equivariant cohomology theory that we want to account for: the Borel equivariant cohomology E^{Bor} associated to any non-equivariant cohomology theory E. Recall that Borel G-equivariant cohomology is defined so that for a G-space X,

$$E^{\mathrm{Bor}}_G(X) \simeq E(X_{\mathrm{h}G}) \simeq E(\mathrm{E}G \times_G X).$$

In particular, for $X = \mathrm{pt}$ the trivial G-space, $E^{\mathrm{Bor}}_G(\mathrm{pt}) \simeq E(\mathrm{B}G)$. Thus we might imagine E^{Bor} as something functorial with respect to the classifying space $\mathrm{B}G$. We will see in §4.3 that the definition motivated by (a) will realize this intuition as well.

4.1.5. Notation. At various points in our discussion of global equivariant homotopy theory, it will be convenient to work with *topological groupoids*, i.e. groupoids internal to the (ordinary) category of topological spaces. These of course include ordinary discrete groupoids. So let's just quickly establish some notation and terminology for these things:

(a) If $\mathscr{G}$ is a topological groupoid, the usual *nerve* construction gives us a simplicial space $\mathrm{N}\mathscr{G}$, and we define the *classifying space* $\mathrm{B}\mathscr{G}$ to be the geometric realization $|\mathrm{N}\mathscr{G}|$ of the nerve.

(b) The most relevant class of examples for us are the *action groupoids*: if X is a topological space with a G-action, there is a topological groupoid $X /\!\!/ G$ whose object space is X and morphism space is $X \times G$, with the structure maps $X \times G \rightrightarrows X$ being the projection and the action. In this case the classifying space $\mathrm{B}(X /\!\!/ G)$ is equivalent to the homotopy quotient $X_{\mathrm{h}G} \simeq \mathrm{E}G \times_G X$. A particularly relevant example in this class comes from the trivial G-space pt, with $\mathrm{B}(\mathrm{pt} /\!\!/ G)$ being the classifying space $\mathrm{B}G \simeq \mathrm{E}G/G$ of the group G. We denote the groupoid $\mathrm{pt} /\!\!/ G$ by $\mathbb{B}G$.

(c) If $\mathscr{G}, \mathscr{H}$ are topological groupoids then there is a topological groupoid $\mathrm{Fun}(\mathscr{G}, \mathscr{H})$ of functors $\mathscr{G} \to \mathscr{H}$. Set-theoretically, $\mathrm{Fun}(\mathscr{G}, \mathscr{H})$ has functors as objects and natural transformations as morphisms; the topologies on these are induced by the relevant mapping space topologies.

(d) We say a (discrete) groupoid $\mathscr{G}$ is *connected* if for any objects $x, y \in \mathscr{G}$ there is a morphism $x \to y$ in $\mathscr{G}$. We say $\mathscr{G}$ is *finite* if for any object $x \in \mathscr{G}$ the automorphism group $\mathrm{Aut}_{\mathscr{G}}(x)$ is finite.

4.1.6. Definition. The *global indexing category* Glo is the 2-category of finite connected (discrete) groupoids; the 1-morphisms are given by functors of groupoids and the 2-morphisms by natural transformations.

Recall that whenever we choose an object of a connected groupoid $\mathscr{G}$ we get an

isomorphism $\mathscr{G}$ with the group $\mathrm{Aut}_{\mathscr{G}}(x)$. So, after choosing basepoints, one can think of objects in Glo simply as the groupoids $\mathbb{B}G$ corresponding to finite groups G.

Just as we warned in (4.1.1), we will for the most part be evil/informal and think of Glo as consisting solely of the objects of the form $\mathbb{B}G$. Then Glo is precisely the 2-category of finite groups described in (4.1.4)(a).

We conclude this subsection by showing how to relate the orbit category $\mathrm{Orb}(G)$ of a fixed group G with the global indexing category Glo. This will allow us in §§4.2 and 4.3 to relate each fixed G-equivariant homotopy theory with global equivariant homotopy theory.

4.1.7. Construction. Let G be a finite group. We construct a fully faithful embedding Θ_G of $\mathrm{Orb}(G)$ into the overcategory $\mathrm{Glo}_{/\mathbb{B}G}$. Here it is convenient *not* to choose basepoints for our objects in $\mathrm{Orb}(G)$ and Glo.

First define a functor $\theta_G\colon \mathrm{Orb}(G) \to \mathrm{Glo}$ by sending $T \in \mathrm{Orb}(G)$ to $\theta_G(T) \coloneqq T /\!\!/ G$, which is connected by definition of an orbit and finite since G is; this is evidently functorial. (If one chooses basepoints, then θ_G can alternatively be thought of as sending $G/H \mapsto \mathbb{B}H$.) Now note that for $\mathrm{pt} \in \mathrm{Orb}(G)$ the trivial orbit, we have $\theta_G(\mathrm{pt}) \simeq \mathbb{B}G$. Since pt is a final object in $\mathrm{Orb}(G)$, the functor θ_G determines a functor

$$\mathrm{Orb}(G) \simeq \mathrm{Orb}(G)_{/\mathrm{pt}} \xrightarrow{\theta_G} \mathrm{Glo}_{/\theta_G(\mathrm{pt})} \simeq \mathrm{Glo}_{/\mathbb{B}G},$$

which we denote Θ_G. That is, Θ_G assigns to $T \in \mathrm{Orb}(G)$ the map $\theta_G(T \to \mathrm{pt})$, which is a map $\theta_G(T) \to \theta_G(\mathrm{pt}) \simeq \mathbb{B}G$. It is straightforward to check that Θ_G is fully faithful.

4.2. The unstable side: global spaces

Now that we've defined the global indexing category, the definition of "space" in global equivariant homotopy theory is immediate.

4.2.1. Definition. A *global space* is a presheaf of spaces on Glo, i.e. a functor $\mathrm{Glo}^{\mathrm{op}} \to \mathrm{Space}$. We denote the ∞-category of global spaces by $\mathrm{Space}_{\mathrm{Glo}}$.

4.2.2. Notation. For G a finite group, we denote the image of $\mathbb{B}G$ under the Yoneda embedding $\mathrm{Glo} \to \mathrm{Space}_{\mathrm{Glo}}$ again by $\mathbb{B}G$. So for each finite group G we have a global space $\mathbb{B}G$, where $\mathbb{B}G(\mathbb{B}H)$ denotes the mapping space $\mathrm{Map}_{\mathrm{Glo}}(\mathbb{B}H, \mathbb{B}G) \simeq \mathrm{B}\,\mathrm{Fun}(\mathbb{B}H, \mathbb{B}G)$.

We claimed in (4.1.4)(a) that this definition of global equivariant homotopy theory would be a useful way to assemble the homotopy theories of G-spaces for all G. We justify this claim now, by embedding, for each finite group G, the homotopy theory of G-spaces into the homotopy theory of global spaces over $\mathbb{B}G$.

4.2.3. Proposition. Let $G \in \mathrm{Glo}$. There is an adjunction

$$\Delta_G\colon G\text{-Space} \rightleftarrows (\mathrm{Space}_{\mathrm{Glo}})_{/\mathbb{B}G} : \Gamma_G,$$

where $(\mathrm{Space}_{\mathrm{Glo}})_{/\mathbb{B}G}$ denotes the overcategory of maps of global spaces $X \to \mathbb{B}G$. Moreover the left adjoint Δ_G is fully faithful.

Proof. In (4.1.7) we constructed a fully faithful embedding $\Theta_G\colon \mathrm{Orb}(G) \hookrightarrow \mathrm{Glo}_{/\mathbb{B}G}$. From this we obtain an adjunction of presheaf categories

$$(\Theta_G)_!\colon \mathrm{Space}_{\mathrm{Orb}(G)} \rightleftarrows \mathrm{Space}_{\mathrm{Glo}_{/\mathbb{B}G}} : (\Theta_G)^*,$$

where $(\Theta_G)^*$ is restriction along Θ_G, and its left adjoint $(\Theta_G)_!$ is left Kan extension along Θ_G (this adjunction is essentially the definition of left Kan extension). Now, by a general fact about the interaction of presheaf categories and overcategories, there

is an equivalence $\mathrm{Space}_{\mathrm{Glo}_{/\mathbb{B}G}} \simeq (\mathrm{Space}_{\mathrm{Glo}})_{/\mathbb{B}G}$. Applying this and the fact/definition (4.1.2) that G-Space $\simeq \mathrm{Space}_{\mathrm{Orb}(G)}$, we obtain the claimed adjunction $\Delta_G \dashv \Gamma_G$. Finally, Θ_G being fully faithful implies immediately that left Kan extension $(\Theta_G)_!$ is fully faithful, and hence Δ_G is fully faithful. □

4.2.4. Remark. It's a straightforward exercise in definition-chasing to unwrap the categorical formalities used in the proof of (4.2.3) and obtain the following more explicit formulae:

(a) Viewing Γ_G as a functor $(\mathrm{Space}_{\mathrm{Glo}})_{/\mathbb{B}G} \to \mathrm{Space}_{\mathrm{Orb}(G)}$, it assigns to a map of global spaces $X \to \mathbb{B}G$ the presheaf $T \mapsto X(T /\!\!/ G) \times_{\mathbb{B}G(T/\!\!/G)} \{\phi\}$, where $\phi\colon T /\!\!/ G \to G$ is the canonical map of groupoids. After choosing basepoints, the presheaf could also be described as $G/H \mapsto X(\mathbb{B}H) \times_{\mathbb{B}G(\mathbb{B}H)} \{i_H\}$, where $i_H\colon H \to G$ is the inclusion of a subgroup, viewed as a map $\mathbb{B}H \to \mathbb{B}G$.

(b) There is a functor $\delta_G\colon G\text{-Top} \to \mathrm{Space}_{\mathrm{Glo}}$ which to $X \in G$-Top assigns the presheaf $\mathscr{H} \mapsto \mathrm{B}\,\mathrm{Fun}(\mathscr{H}, X /\!\!/ G)$. In the case $\mathscr{H} = \mathbb{B}H$, note that $\mathrm{Fun}(\mathbb{B}H, X /\!\!/ G)$ is simply the action groupoid

$$\left(\coprod_{\phi \in \mathrm{Hom}(H,G)} X^{\mathrm{im}(\phi)}\right) /\!\!/ G.$$

E.g. $\delta_G(\mathrm{pt}) \simeq \mathbb{B}G$. Then, viewing Δ_G as a functor $G\text{-Top} \to (\mathrm{Space}_{\mathrm{Glo}})_{/\mathbb{B}G}$, it assigns to $X \in G$-Top the map $\delta_G(X \to \mathrm{pt})$, which is a map $\delta_G(X) \to \delta_G(\mathrm{pt}) \simeq \mathbb{B}G$.

That's all we have to say about the general theory of global spaces, since that's all that will be relevant to our study of character theory. See [11] for some more interesting phenomena in this setting.

4.3. The stable side: global spectra

In equivariant homotopy theory, the stable side is a bit more subtle than the unstable side. For a fixed group G, there is a distinction made between what are known as *genuine* G-spectra and what are sometimes called *naïve* G-spectra. A similar subtlety exists in the world of global equivariant homotopy theory, but in this thesis we will be happy to ignore it. That is, we will work exclusively with naïve G-spectra and global spectra, although we will omit the modifier from our terminology for the sake of our self-esteem.

We first recall the definition for a fixed finite group G.

4.3.1. Definition. A *G-spectrum* is a presheaf of spectra on $\mathrm{Orb}(G)$, i.e. a functor $\mathrm{Orb}(G)^{\mathrm{op}} \to \mathrm{Spect}$. We denote the ∞-category of G-spectra both by $\mathrm{Spect}_{\mathrm{Orb}(G)}$ and by G-Spect.

4.3.2. Remark. Since G-Space $\simeq \mathrm{Space}_{\mathrm{Orb}(G)}$ is the free cocompletion of $\mathrm{Orb}(G)$, one can equivalently view a G-spectrum as a functor $E\colon G\text{-Space}^{\mathrm{op}} \to \mathrm{Spect}$ which takes colimits of G-spaces to limits of spectra. We will generally view G-spectra in this way without warning, but it's quite natural: the spectrum $E(X)$ encodes the E-cohomology of $X \in G$-Space, in that its homotopy groups give the cohomology groups $E^*(X)$ when we think of E as a G-equivariant cohomology theory.

We now make the analagous definition for the global case.

4.3.3. Definition. A *global spectrum* is a presheaf of spectra on Glo, i.e. a functor $\mathrm{Glo}^{\mathrm{op}} \to \mathrm{Spect}$. We denote the ∞-category of global spaces by $\mathrm{Spect}_{\mathrm{Glo}}$.

4.3.4. Remark. Analagously to (4.3.2), one can equivalently view a global spectrum as a functor $E\colon \mathrm{Space}_{\mathrm{Glo}}^{\mathrm{op}} \to \mathrm{Spect}$ which takes colimits of global spaces to limits of spectra. Again, we will view global spectra in this way without warning, and the spectrum $E(X)$ encodes the E-cohomology of $X \in \mathrm{Space}_{\mathrm{Glo}}$, in that it's homotopy groups give the cohomology groups $E^*(X)$ when we think of E as a global equivariant cohomology theory.

4.3.5. Example. As remarked in (b), given a spectrum $E \in \mathrm{Spect}$, we expect the Borel-equivariant cohomology theories associated to E to assemble into a global spectrum. This is indeed the case: we define E^{Bor} to be the global spectrum given by the presheaf $\mathbb{B}G \mapsto E(\mathrm{B}G)$, where $E(\mathrm{B}G)$ denotes the function spectrum as it has earlier. When it's clear from context what we mean, we will often just write E in place of E^{Bor}.

Seeing that this is indeed a functor $\mathrm{Glo}^{\mathrm{op}} \to \mathrm{Spect}$ is fairly straightforward, but it's interesting to note that we can obtain E^{Bor} by a purely formal procedure in this framework. Namely, Consider the trivial subcategory $\{\mathbb{B}1\}$ of Glo, where 1 denotes the trivial group. Then E can tautologically be viewed as a functor $\{\mathbb{B}1\}^{\mathrm{op}} \to \mathrm{Spect}$, and E^{Bor} is just the right Kan extension of E along the inclusion $\{\mathbb{B}1\} \hookrightarrow \mathrm{Glo}$. This just amounts to saying that for a finite group G, the mapping space $\mathrm{Map}_{\mathrm{Glo}}(\mathbb{B}1, \mathbb{B}G)$ is equivalent to $\mathrm{B}G$, and this is evident from the definition.

4.3.6. The essential point of defining global equivariant cohomology theories was that they should package together nicely related G-equivariant cohomology theories for all finite groups G. So we should say how a global spectrum $E \in \mathrm{Spect}_{\mathrm{Glo}}$ determines a G-spectrum $E_G \in G\text{-Spect}$ for each G. It's easy to guess how this goes: recall that we have an embedding $\Theta_G\colon \mathrm{Orb}(G) \to \mathrm{Glo}_{/\mathbb{B}G}$, underlying which is a functor $\theta_G\colon \mathrm{Orb}(G) \to \mathrm{Glo}$. Restriction in θ_G determines our functor

$$\mathrm{Spect}_{\mathrm{Glo}} \to \mathrm{Spect}_{\mathrm{Orb}(G)} \simeq G\text{-Spect},$$

which we denote $E \mapsto E_G$.

As an example, let E be a spectrum and consider the global spectrum E^{Bor} defined in (4.3.5). By definition of θ_G, the G-spectrum E_G^{Bor} is given by the presheaf $G/H \mapsto E(\mathrm{B}H)$. This is indeed the presheaf corresponding to the Borel G-equivariant cohomology theory assoicated to E, which on a general G-space X is given by $E_G(X) \simeq E(X_{\mathrm{h}G})$, where $X_{\mathrm{h}G} := \mathrm{E}G \times_G X$ is the homotopy quotient.

4.3.7. Notation. Let A be an E_∞-ring. Suppose we have a global spectrum $E \in \mathrm{Spect}_{\mathrm{Glo}}$ which takes values in A-modules, i.e. which factors through the forgetful functor from the ∞-category of A-modules to Spect. Then given an A-module B, we may form a new global spectrum $B \otimes_A E$, defined by

$$(B \otimes_A E)(\mathbb{B}G) := B \otimes_A E(\mathbb{B}G)$$

for all finite groups G (which is evidently functorial). When A is understood we will often just denote this global spectrum by $B \otimes E$.

5. Abelian descent

In many cases of interest, a global spectrum is determined by its values on abelian groups. This section is devoted to understanding this phenomenon, which can be encoded rigorously as follows.

5.0.1. Notation. Let Alo denote the full subcategory of Glo spanned by the (finite) abelian groups.

5.0.2. Definition. We say a global spectrum $E \in \mathrm{Spect}_{\mathrm{Glo}}$ satisfies *abelian descent* if it is the right Kan extension of its restriction to Alo, i.e. if the canonical map

$$E(\mathbb{B}G) \longrightarrow \lim_{\mathbb{B}A \in \mathrm{Alo}_{/\mathbb{B}G}} E(\mathbb{B}A)$$

is an equivalence for all finite groups G.

5.1. Alternate characterizations

In this short subsection we unpack the definition (5.0.2) of abelian descent by giving a tautological reformulation, and then giving (again fairly tautological) reformulations of this reformulation. But all this tautology is worth something! It will allow us to give (in the next subsection) a more concrete way to check that a global spectrum satisfies abelian descent, so just stay tuned for a moment.

5.1.1. Definition. For $X \in \mathrm{Space}_{\mathrm{Glo}}$ we define the *abelianization* X^{ab} of X by

$$X^{\mathrm{ab}} := \operatorname*{colim}_{\mathbb{B}A \in \mathrm{Alo}_{/X}} \mathbb{B}A \in \mathrm{Space}_{\mathrm{Glo}}.$$

Note that there is a canonical map $X^{\mathrm{ab}} \longrightarrow X$. To save some parentheses, we'll write $\mathbb{B}^{\mathrm{ab}}G$ in place of $(\mathbb{B}G)^{\mathrm{ab}}$ for $G \in \mathrm{Glo}$.

5.1.2. Lemma. A global spectrum $E \in \mathrm{Spect}_{\mathrm{Glo}}$ satisfies abelian descent if and only if the canonical map $E(\mathbb{B}G) \longrightarrow E(\mathbb{B}^{\mathrm{ab}}G)$ is an equivalence for all $G \in \mathrm{Glo}$.

Proof. This is immediate from the definitions (5.0.2, 5.1.1), as E takes colimits in $\mathrm{Space}_{\mathrm{Glo}}$ to limits in Spect. □

5.1.3. Lemma. Let $\mathrm{ab}\colon \mathrm{Glo} \longrightarrow \mathrm{Alo}$ be the functor sending a group $\mathbb{B}H$ to its group-theoretic abelianization[15] $\mathbb{B}H^{\mathrm{ab}}$. For $X \in \mathrm{Space}_{\mathrm{Glo}}$ we have $X^{\mathrm{ab}} \simeq X \circ \mathrm{ab}$, i.e. there are functorial equivalences $X^{\mathrm{ab}}(\mathbb{B}H) \simeq X(\mathbb{B}H^{\mathrm{ab}})$ for $H \in \mathrm{Glo}$.

Proof. Let $X \in \mathrm{Space}_{\mathrm{Glo}}$ and $\mathbb{B}H \in \mathrm{Glo}$. Observe that for any $\mathbb{B}A \in \mathrm{Alo}$ the canonical map $\mathbb{B}A(\mathbb{B}H^{\mathrm{ab}}) \longrightarrow \mathbb{B}A(\mathbb{B}H)$ is an equivalence. It follows that we have an equivalence

$$X^{\mathrm{ab}}(\mathbb{B}H) \simeq \operatorname*{colim}_{\mathbb{B}A \in \mathrm{Alo}_{/X}} \mathbb{B}A(\mathbb{B}H) \simeq \operatorname*{colim}_{\mathbb{B}A \in \mathrm{Alo}_{/X}} \mathbb{B}A(\mathbb{B}H^{\mathrm{ab}}),$$

functorial in $\mathbb{B}H$. But now the canonical map

$$\operatorname*{colim}_{\mathbb{B}A \in \mathrm{Alo}_{/X}} \mathbb{B}A(\mathbb{B}H^{\mathrm{ab}}) \longrightarrow X(\mathbb{B}H^{\mathrm{ab}})$$

is clearly an equivalence since $\mathbb{B}H^{\mathrm{ab}} \in \mathrm{Alo}$. □

5.1.4. Lemma. Let G be a finite group. Let $\Delta_G \dashv \Gamma_G$ be the adjunction constructed in (4.2.3).

(a) For a subgroup $H \subseteq G$, the fixed-point space $\Gamma_G(\mathbb{B}^{\mathrm{ab}}G)^H$ is contractible if H is abelian and empty if H is nonabelian.

(b) Conversely, if X is a G-space such that X^H is contractible if H is abelian and empty if H is nonabelian, then $\Delta_G(X)$ is equivalent to $\mathbb{B}^{\mathrm{ab}}G \longrightarrow \mathbb{B}G$ in $(\mathrm{Space}_{\mathrm{Glo}})_{/\mathbb{B}G}$.

Proof. Take a subgroup $H \subseteq G$. By (4.2.4)(a), $\Gamma_G(\mathbb{B}^{\mathrm{ab}}G)^H$ is given by

$$\mathbb{B}^{\mathrm{ab}}G(\mathbb{B}H) \times_{\mathbb{B}G(\mathbb{B}H)} \{i_H\} \simeq \mathbb{B}G(\mathbb{B}H^{\mathrm{ab}}) \times_{\mathbb{B}G(\mathbb{B}H)} \{i_H\},$$

where $i_H \in \mathbb{B}G(H)$ corresponds to the inclusion $H \hookrightarrow G$; this clearly is contractible when H is abelian and empty when H is nonabelian, proving (a).

[15] Recall the abelianization of a group H is the quotient by its commutator subgroup $H/[H, H]$.

Now let X be a G-space with this same property: X^H is contractible for H abelian and empty for H nonabelian. Recall $\Delta_G(X)$ is the canonical map $\delta_G(X) \to \delta_G(*) \simeq \mathbb{B}G$, where for a G-space Y, $\delta_G(Y)(\mathbb{B}H)$ is the homotopy quotient

$$\left(\coprod_{\phi \in \mathrm{Hom}(H,G)} Y^{\mathrm{im}(\phi)}\right)_{\mathrm{h}G} \simeq \mathrm{B}\left(\left(\coprod_{\phi \in \mathrm{Hom}(H,G)} Y^{\mathrm{im}(\phi)}\right) /\!\!/ G\right)$$

for $\mathbb{B}H \in \mathrm{Glo}$. Since $X^{\mathrm{im}(\phi)}$ is only nonempty when $\mathrm{im}(\phi)$ is abelian, i.e. when ϕ factors through H^{ab}, we have

$$\delta_G(X)(H) \simeq \left(\coprod_{\phi \in \mathrm{Hom}(H^{\mathrm{ab}},G)} X^{\mathrm{im}(\phi)}\right)_{\mathrm{h}G}.$$

It follows from (5.1.3) that $\delta_G(X) \to \mathbb{B}G$ factors through the map $\mathbb{B}^{\mathrm{ab}}G \to \mathbb{B}G$; for $\mathbb{B}H \in \mathrm{Glo}$ the resulting map $\delta_G(X) \to \mathbb{B}^{\mathrm{ab}}G$ looks like

$$\left(\coprod_{\phi \in \mathrm{Hom}(H^{\mathrm{ab}},G)} X^{\mathrm{im}(\phi)}\right)_{\mathrm{h}G} \simeq \delta_G(X)(H) \to \mathbb{B}^{\mathrm{ab}}G(H) \simeq \left(\coprod_{\phi \in \mathrm{Hom}(H^{\mathrm{ab}},G)} \mathrm{pt}\right)_{\mathrm{h}G},$$

induced by the unique maps $X^{\mathrm{im}(\phi)} \to \mathrm{pt}$. But these maps are all equivalences by our contracibility hypothesis, whence $\delta_G(X) \to \mathbb{B}^{\mathrm{ab}}G$ is an equivalence (over $\mathbb{B}G$), as desired. □

5.2. Complex-oriented descent

The takeaway of §5.1 is that whether or not a global spectrum satisfies abelian descent comes down to comparing, for each finite group G, what it assigns to the trivial G-space and what it assigns to a G-space whose H-fixed point space is contractible for $H \subseteq G$ abelian and empty for $H \subseteq G$ nonabelian. In this subsection we construct a more explicit model for the latter, and use this to describe a practical method for proving that various global spectra associated to complex-oriented cohomology theories satisfy abelian descent. This idea is originally due to Quillen [9], and is key to our approach to character theory.

5.2.1. Notation. For the remainder of this subsection:

(a) Fix a finite group G.

(b) Fix a faithful complex representation[16] $\rho\colon G \hookrightarrow \mathrm{Aut}(V)$.

(c) Choosing a G-equivariant hermitian inner product[17] on and a basis of V, this determines an embedding into the unitary group $i\colon G \hookrightarrow \mathrm{U}(k)$ for $k := \dim(V)$.

(d) Let $T \subseteq \mathrm{U}(k)$ be a maximal torus. Let $F := \mathrm{U}(k)/T$, which is a G-space via i. Recall that F is the space of complete flags in V (if you imagine T as the subspace of unitary diagonal matrices, then F is evidently the space of k-tuples of orthogonal lines in V, which is the same as the space of complete flags).

5.2.2. Lemma. For a subgroup $H \subseteq G$, the fixed-point space F^H is nonempty if and only if H is abelian.

Proof. This is immediate from the theory of maximal tori. It's clear that F^H is nonempty if and only if some conjugage uHu^{-1} is contained in our maximal torus T, which is true if and only if H is abelian. □

[16] Recall that such a thing always exists, e.g. the regular representation $\mathbb{C}[G]$ always works.

[17] Recall that such a thing always exists, e.g. by averaging any ordinary hermitian inner product over G.

5.2.3. Notation. If X is a space, let $\mathrm{E}X$ denote the geometric realization of the simplicial space

$$X \Leftarrow X \times X \Lleftarrow X \times X \times X \Llleftarrow \cdots .$$

5.2.4. Lemma. Let X be a space. If X is empty then $\mathrm{E}X$ is empty; if X is nonempty then $\mathrm{E}X$ is contractible.

Proof. The claim is obvious for X empty. For X nonempty there is a standard way to construct contracting homotopies (see e.g. [3, 3.14]). □

5.2.5. Lemma. For a subgroup $H \subseteq G$, the fixed-point space $(\mathrm{E}F)^H$ is contractible for H abelian and empty for H nonabelian. In other words, by (5.1.4), $\mathrm{E}F$ is a model for $\mathbb{B}^{\mathrm{ab}}G$.

Proof. Since geometric realization of simplicial spaces commutes with finite limits, in particular taking H-fixed-points, this is immediate from (5.2.2, 5.2.4). □

5.2.6. Lemma. Let E_G be a ring G-spectrum. Suppose $E_G^* \to E_G^*(F)$ is a faithfully flat map of rings, and that the canonical map $E_G^*(F)^{\otimes n} \to E_G^*(F^{\times n})$ is an isomorphism for all $n \geq 1$. Then the canonical map $E_G^* \to E_G^*(\mathrm{E}F)$ is an isomorphism.

Proof. Consider the function G-spectrum $E_G(\mathrm{E}F)$. Since $\mathrm{E}F$ is defined by a geometric realization, this function spectrum is the the totalization of the cosimplicial G-spectrum

$$E_G(F) \rightrightarrows E_G(F \times F) \Rrightarrow E_G(F \times F \times F) \Rrrightarrow \cdots .$$

Now, the cohomology ring $E_G^*(\mathrm{E}F)$ is given by the homotopy groups of the G-fixed points $E_G(\mathrm{E}F)^G$ of this function spectrum. Viewing G-spectra as presheaves of spectra on $\mathrm{Orb}(G)$, taking G-fixed points amounts to evaluating on the orbit G, and so we see that $E_G(\mathrm{E}F)^G$ is the totalization of the cosimplicial spectrum

$$E_G(F)^G \rightrightarrows E_G(F \times F)^G \Rrightarrow E_G(F \times F \times F)^G \Rrrightarrow \cdots .$$

We apply the Bousfield-Kan spectral sequence arising from this cosimplicial spectrum, which will converge to the homotopy groups of the totalization, i.e. $E_G^*(\mathrm{E}F)$, and which has second page given by the complex

$$E_G^*(F) \rightrightarrows E_G^*(F \times F)) \Rrightarrow E_G^*(F \times F \times F) \Rrrightarrow \cdots .$$

By hypothesis we have $E_G^*(F^n) \simeq E_G^*(F)^{\otimes n}$, so this is just the Amitsur complex of the map $E_G^* \to E_G^*(F)$. But since this map is faithfully flat, the homology of this complex is just E_G^* concentrated in degree 0. It follows that the spectral sequence degenerates and gives the desired isomorphism $E_G^* \xrightarrow{\sim} E_G^*(\mathrm{E}F)$. □

5.2.7. Lemma ([5, §2.2–2.3]). Let E be a complex-orientable ring spectrum.

(a) Let $\mathscr{V} \to X$ be a vector bundle, and $\mathscr{F} \to X$ the associated bundle of complete flags in $\mathscr{V}$. Then $E^*(\mathscr{F})$ is a finite free module over $E^*(X)$.

(b) Moreover, suppose we have a map $Y \to X$. Let $\mathscr{V}_Y$ be the pullback of $\mathscr{V}$ to Y and $\mathscr{F}_Y$ the bundle of complete flags in $\mathscr{V}_Y$. Then the canonical map

$$E^*(Y) \otimes_{E^*(X)} E^*(\mathscr{F}) \to E^*(\mathscr{F}_Y)$$

is an isomorphism.

(c) Let E_G be the ring G-spectrum representing the Borel G-equivariant cohomology associated to E. Then $E_G^*(F)$ is a finite free module over E_G^*, and the canonical map $E_G^*(F)^{\otimes n} \to E_G^*(F^{\times n})$ is an isomorphism for all $n \geq 1$.

Sketch of proof. Recall that E_G is defined such that $E_G^*(F) \simeq E^*(\mathrm{E}G \times_G F)$ and $E_G^* \simeq E^*(\mathrm{B}G)$. Noting that $\mathrm{E}G \times_G F \to BG$ is the bundle of complete flags in the vector bundle $EG \times_G V \to BG$, we see that the first assertion of (c) follows from (a). The second then follows from (b), where we inductively pull back via the map $\mathrm{E}G \times_G F^{\times n} \to \mathrm{E}G \times_G F^{\times(n-1)}$.

We prove (a) by induction on the rank of the vector bundle $\mathscr{V} \to X$. Of course the claim is trivial when the rank is 0, giving us our base case, so assume the rank is $n \geq 1$. Consider the associated projective bundle $\mathbb{P}\mathscr{V} \to X$. The pullback bundle $\mathscr{V}' := \mathbb{P}\mathscr{V} \times_X \mathscr{V}$ over $\mathbb{P}V$ contains a canonical line sub-bundle $\mathscr{L}$. Contemplating the quotient bundle $\mathscr{W} := \mathscr{V}'/\mathscr{L}$, we see that the space of complete flags in $\mathscr{W}$ can be identified with the space $\mathscr{F}$ of complete flags in $\mathscr{V}$. Thus the map $E^*(X) \to E^*(\mathscr{F})$ factors as the composition $E^*(X) \to E^*(\mathbb{P}\mathscr{V}) \to E^*(\mathscr{F})$. Since $\mathscr{W} \to \mathbb{P}\mathscr{V}$ has rank $n-1$, we know by induction that $E^*(\mathscr{F})$ is a finite free $E^*(\mathbb{P}\mathscr{V})$-module. So we just need that $E^*(\mathbb{P}\mathscr{V})$ is a finite free $E^*(X)$-module. This is a standard application of the Leray-Hirsch theorem, the point being that $\mathbb{P}V \to X$ is a bundle with fiber $\mathbb{C}\mathbb{P}^n$ and a complex orientation of E induces an isomorphism $E^*(\mathbb{C}\mathbb{P}^n) \simeq E^*[x]/(x^n)$, which of course is finite free. Statement (b) will then follow from the naturality of this argument. □

Our work in this section culminates in the following result.

5.2.8. Proposition. Let E be a complex-orientable E_∞-ring spectrum. Let $E \to C$ be a flat map of E_∞-rings. Then the global spectrum $C \otimes E \in \mathrm{Spect}_{\mathrm{Glo}}$ given by the presheaf $\mathbb{B}G \mapsto C \otimes_E E(\mathrm{B}G)$ satisfies abelian descent.

Proof. By (5.1.2) it suffices to show $(C \otimes E)(\mathbb{B}G) \to (C \otimes E)(\mathbb{B}^{\mathrm{ab}}G)$ is an equivalence for any fixed finite group G. By (5.1.4, 5.2.5) it suffice to show that the map $(C \otimes E)_G^* \to (C \otimes E)_G^*(\mathrm{E}F)$ is an isomorphism. By (5.2.6) it suffices to show that $(C \otimes E)_G^* \to (C \otimes E)_G^*(F)$ is faithfully flat and that $(C \otimes E)_G^*(F)^{\otimes n} \to (C \otimes E)_G^*(F^{\times n})$ is an isomorphism for all $n \geq 1$. Since F is a compact G-manifold, hence equivalent to a finite G-space, and C is flat over E, these maps are just given by $C^* \otimes_{E^*} E_G^* \to C^* \otimes_{E^*} E_G^*(F)$ and $C_* \otimes_{E^*} E_G^*(F)^{\otimes n} \to C^* \otimes_{E^*} E_G^*(F^{\times n})$, respectively. We are then done by (5.2.7)(c). □

6. Character theory

We finally arrive at the main course of this thesis. In §6.1 we define our p-adic loop space functor, and in §6.2, everything comes together at once to prove our generalized character theory.

6.0.1. Notation. Throughout this section we work at a fixed prime p.

6.1. Loops

6.1.1. Notation. There is an internal hom in the ∞-category $\mathrm{Space}_{\mathrm{Glo}}$, which we denote $[-,-]$. It satisfies the adjunction

$$\mathrm{Map}_{\mathrm{Space}_{\mathrm{Glo}}}(X \times Y, Z) \simeq \mathrm{Map}_{\mathrm{Space}_{\mathrm{Glo}}}(X, [Y, Z]),$$

which means it can be computed via the formula

$$[Y, Z](\mathbb{B}G) \simeq \mathrm{Map}_{\mathrm{Space}_{\mathrm{Glo}}}(\mathbb{B}G, [Y, Z]) \simeq \mathrm{Map}_{\mathrm{Space}_{\mathrm{Glo}}}(\mathbb{B}G \times Y, Z).$$

In particular, for $Y = \mathbb{B}H$ we have

$$
\begin{aligned}
[\mathbb{B}H, Z](\mathbb{B}G) &\simeq \mathrm{Map}_{\mathrm{Space}_{\mathrm{Glo}}}(\mathbb{B}G \times \mathbb{B}H, Z) \\
&\simeq \mathrm{Map}_{\mathrm{Space}_{\mathrm{Glo}}}(\mathbb{B}(G \times H), Z) \\
&\simeq Z(\mathbb{B}(G \times H))
\end{aligned} \tag{6.1.2}
$$

6.1.3. Definition. We define the *(p-adic) loop space* functor:

$$
\mathscr{L} \colon \mathrm{Space}_{\mathrm{Glo}} \to \mathrm{Space}_{\mathrm{Glo}}, \quad X \mapsto \operatorname*{colim}_k [\mathbb{B}\mathbb{Z}/p^k, X].
$$

One may want to think of this as an internal hom $[\mathbb{B}\mathbb{Z}_p, -]$ from the pro-object $\mathbb{B}\mathbb{Z}_p \coloneqq \lim_k \mathbb{B}\mathbb{Z}/p^k$ in $\mathrm{Space}_{\mathrm{Glo}}$.

6.1.4. Lemma. The loop space functor $\mathscr{L}$ preserves colimits.

Proof. Since colimits commute with other colimits, it suffices to show for each k that $[\mathbb{B}\mathbb{Z}/p^k, -]$ preserves colimits, which is easy to see: by (6.1.2),

$$
\begin{aligned}
[\mathbb{B}\mathbb{Z}/p^k, \operatorname*{colim}_i X_i](\mathbb{B}G) &\simeq \left(\operatorname*{colim}_i X_i\right)(\mathbb{B}(G \times \mathbb{Z}/p^k)) \\
&\simeq \operatorname*{colim}_i X_i(\mathbb{B}(G \times \mathbb{Z}/p^k)) \\
&\simeq \operatorname*{colim}_i [\mathbb{B}\mathbb{Z}/p^k, X_i](\mathbb{B}G).
\end{aligned}
$$

□

6.1.5. Lemma. Let G be a finite group and let X be a G-space, which we will abusively identify with its associated global space via the functor $\delta_G \colon G\text{-Space} \to \mathrm{Space}_{\mathrm{Glo}}$ defined in (4.2.4). Then its loop space $\mathscr{L}X$ is the global space associated to the G-space

$$
Y \coloneqq \coprod_{\alpha \in \mathrm{Hom}(\mathbb{Z}_p, G)} X^{\mathrm{im}(\alpha)}.
$$

Proof. It suffices to prove the claim after replacing $\mathscr{L}X$ with $[\mathbb{B}\mathbb{Z}/p^k, X]$ and Y with $Y_k \coloneqq \coprod_{\alpha \in \mathrm{Hom}(\mathbb{Z}/p^k, G)} X^{\mathrm{im}(\alpha)}$, since G being finite implies $Y \simeq Y_k$ for sufficiently large k. When we view Y_k as a global space, $Y_k(\mathbb{B}H)$ is the classifying space of the action groupoid

$$
\left(\coprod_{\beta \in \mathrm{Hom}(H, G)} Y_k^{\mathrm{im}(\beta)}\right) /\!\!/ G.
$$

Now, for $\alpha \in \mathrm{Hom}(\mathbb{Z}/p^k, G)$ and $\beta \in \mathrm{Hom}(H, G)$, $x \in X^{\mathrm{im}(\alpha)} \subseteq Y_k$ is fixed by $\mathrm{im}(\beta)$ if and only if $\mathrm{im}(\alpha)$ and $\mathrm{im}(\beta)$ commute within G and x is a fixed point of the group $\mathrm{im}(\alpha)\,\mathrm{im}(\beta)$ they generate. So this action groupoid can be rewritten as

$$
\left(\coprod_{\gamma \in \mathrm{Hom}(H \times \mathbb{Z}/p^k)} X^{\mathrm{im}(\gamma)}\right) /\!\!/ G.
$$

The classifying space of this is by definition equivalent to $X(\mathbb{B}(H \times \mathbb{Z}/p^k))$, which by (6.1.2) is equivalent to $[\mathbb{B}\mathbb{Z}/p^k, X](\mathbb{B}H)$. So we conclude $Y_k \simeq [\mathbb{B}\mathbb{Z}/p^k, X]$, as desired. □

6.1.6. Lemma. Let G be a finite group. We have

$$
\mathscr{L}\mathbb{B}G \simeq \coprod_{[\alpha]} \mathbb{B}C(\alpha),
$$

where $[\alpha]$ runs over the conjugacy classes of homomorphisms $\mathbb{Z}/p^k \to G$ for some sufficiently large k (the largest p-power torsion in G) and $C(\alpha) \subseteq G$ denotes the

centralizer of α. In particular, for $G = A$ abelian there is a natural equivalence

$$\mathscr{L}\mathbb{B}A \simeq \coprod_{a \in A[p^\infty]} \mathbb{B}A,$$

where $A[p^\infty] \subseteq A$ is the subgroup of p-power torsion.

Proof. This follows immediately from (6.1.5). □

6.1.7. Lemma. Let E be a global spectrum. Then:

(a) The composite $E' := E \circ \mathscr{L} \colon \mathrm{Space}_{\mathrm{Glo}}^{\mathrm{op}} \to \mathrm{Spect}$ is also a global spectrum, i.e. takes colimits of global spaces to limits of spectra.

(b) If E satisfies abelian descent, then so does E'.

Proof. Statement (a) is immediate from $\mathscr{L}$ being colimit-preserving (6.1.4). For the second statement, we would like to prove that $E'(\mathbb{B}G) \to E'(\mathbb{B}^{\mathrm{ab}}G)$ is an equivalence for all G, assuming this holds for E. We first observe that for X a global space, there's a natural equivalence of global spaces $(\mathscr{L}X)^{\mathrm{ab}} \simeq \mathscr{L}X^{\mathrm{ab}}$:

$$\begin{aligned}(\mathscr{L}X)^{\mathrm{ab}}(\mathbb{B}H) &\simeq \mathscr{L}X(\mathbb{B}H^{\mathrm{ab}}) \\ &\simeq \operatorname*{colim}_k X(\mathbb{B}(H^{\mathrm{ab}} \times \mathbb{Z}/p^k)) \\ &\simeq \operatorname*{colim}_k X(\mathbb{B}(H \times \mathbb{Z}/p^k)^{\mathrm{ab}}) \\ &\simeq \mathscr{L}X^{\mathrm{ab}}(\mathbb{B}H),\end{aligned}$$

where we have applied (5.1.2, 6.1.2). Combining this with (6.1.6), we see that

$$\mathscr{L}(\mathbb{B}^{\mathrm{ab}}G) \simeq (\mathscr{L}\mathbb{B}G)^{\mathrm{ab}} \simeq \left(\coprod_{[\alpha]} \mathbb{B}C(\alpha)\right)^{\mathrm{ab}},$$

and by (5.1.2) the last of these is equivalent to $\coprod_{[\alpha]} \mathbb{B}^{\mathrm{ab}}C(\alpha)$. Finally, after chasing these equivalences we see that the map $E'(\mathbb{B}G) \to E'(\mathbb{B}^{\mathrm{ab}}G)$ is the map

$$E(\mathscr{L}\mathbb{B}G) \simeq E\left(\coprod_{[\alpha]} \mathbb{B}C(\alpha)\right) \to E\left(\coprod_{[\alpha]} \mathbb{B}^{\mathrm{ab}}C(\alpha)\right) \simeq E(\mathscr{L}\mathbb{B}^{\mathrm{ab}}G),$$

induced by the coproduct of the canonical maps $\mathbb{B}^{\mathrm{ab}}C(\alpha) \to \mathbb{B}C(\alpha)$. Since E takes coproducts to products, it follows that if E satisfies abelian descent then so does E'. □

6.2. The main theorem

We are now ready to state and prove our version of character theory.

6.2.1. Notation. Let E be a Morava E-theory of height n, and let $L_t := L_{K(t)}E$. Let C_t be the flat L_t-algebra constructed in (3.3.15). We will denote the associated Borel global spectra also by E and L_t.

6.2.2. By (6.1.7) we have a global spectrum $L_t \circ \mathscr{L}^{n-t}$, which by (6.1.6) is given by the formula

$$\mathbb{B}G \mapsto L_t\left(\coprod_{[\alpha]} \mathrm{B}C(\alpha)\right) \simeq \bigoplus_{[\alpha]} L_t(\mathrm{B}C(\alpha)), \tag{6.2.3}$$

where now $[\alpha]$ runs over conjugacy classes of homomorphisms $(\mathbb{Z}/p^k)^{n-t} \to G$ (for sufficiently large k). So $L_t \circ \mathscr{L}^{n-t}$ again takes values in L_t-modules, and thus we may

define $C_t \otimes (L_t \circ \mathscr{L}^{n-t})$. And since tensor product commutes with direct sum, this is equivalent to the global spectrum $(C_t \otimes L_t) \circ \mathscr{L}^{n-t}$.

6.2.4. Theorem. There is an equivalence of global spectra

$$C_t \otimes E \simeq C_t \otimes (L_t \circ \mathscr{L}^{n-t}).$$

Proof. By (5.2.8) the global spectra $C_t \otimes E$ and $C_t \otimes L_t$ satisfy abelian descent. By (6.1.7) this implies $C_t \otimes (L_t \circ \mathscr{L}^{n-t}) \simeq (C_t \otimes L_t) \circ \mathscr{L}^{n-t}$ also satisfies abelian descent. Thus is suffices to give the desired equivalence after restricting from Glo to Alo. For finite abelian groups A, the formula (6.2.3), coming from (6.1.6), gives a natural equivalence

$$(C_t \otimes (L_t \circ \mathscr{L}^{n-t}))(\mathbb{B}A) \simeq C_t \otimes_{L_t} \bigoplus_{(A[p^\infty])^{n-t}} L_t(\mathrm{B}A),$$

which is naturally equivalent to $(C_t \otimes E)(\mathbb{B}A) \simeq C_t \otimes_A E(\mathrm{B}A)$ by our analysis of p-divisible groups (3.3.17). □

6.2.5. Remark. Since C_t is flat over L_t and E by (3.3.16), our theorem gives for each finite group G an equivalence of G-equivariant cohomology theories

$$C_t^* \otimes_{E^*} E_G(X) \simeq C_t^* \otimes_{L_t^*} L_t^*(\mathscr{L}^{n-t}X),$$

for finite G-spaces X. So we indeed recover isomorphisms of the form proven in [5, 13]. Note however that we did not explicitly construct this isomorphism when G is nonabelian.

And with that have completed our mission! To conclude, let me just summarize the ideas that got us here. Complex-oriented descent allowed us to show that the the global spectra $C_t \otimes E$ and $C_t \otimes (L_t \circ \mathscr{L}^{n-t})$ are determined by their values on abelian groups, and hence in essence by the p-divisible groups $\mathbb{G}_E$ and $\mathbb{G}_{L_t}$ associated to Morava E-theory and its localization. Then the fact that we could identify the two sides of character theory simply boiled down to the behavior of these p-divisible groups: when $\mathbb{G}_E$ is base changed to L_t, it loses $n-t$ infinitesimal height, and gains $n-t$ étale height; then when base changed to C_t, this étale part becomes constant, which translates into the $(n-t)$-fold looping on the right-hand side.

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