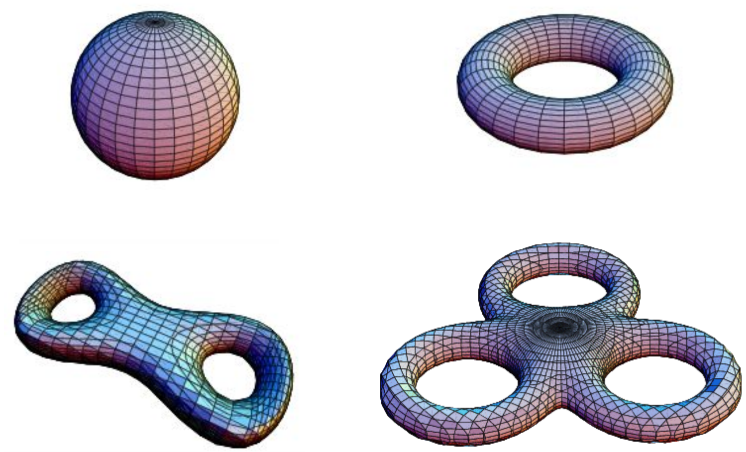
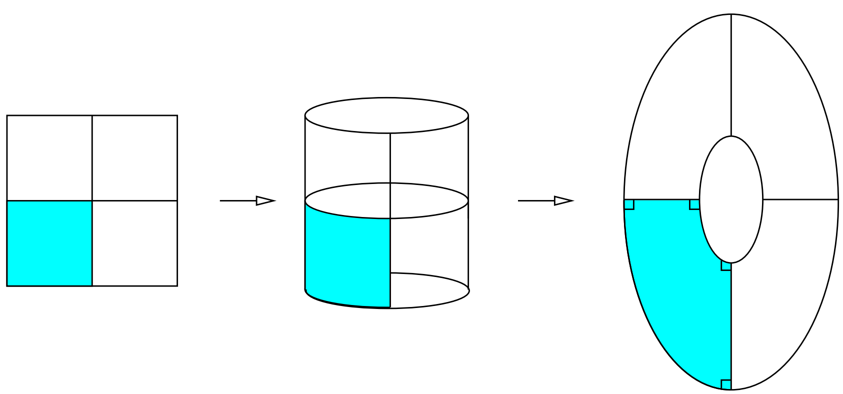
The Evolution of Geometric Structures on 3-Manifolds

Curtis T McMullen
Harvard University

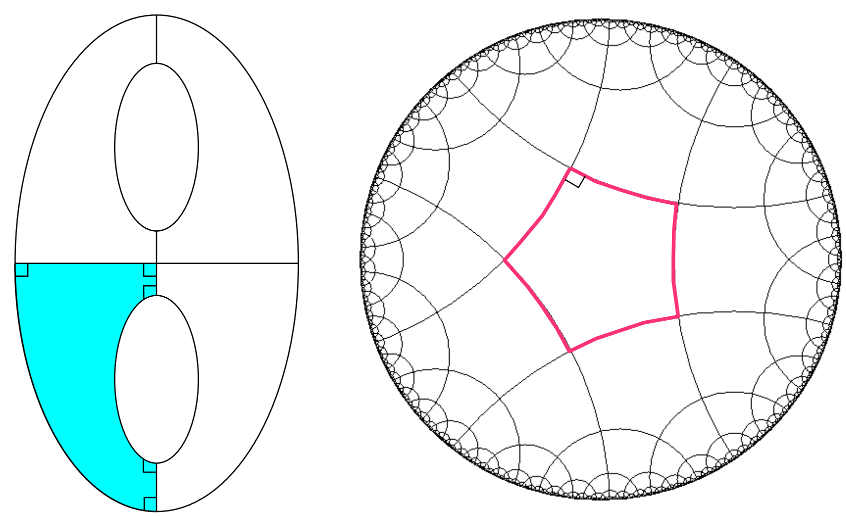
Surfaces of genus 0, 1, 2, 3



Squares tile the torus



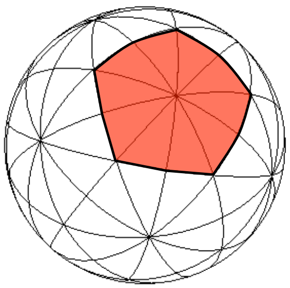
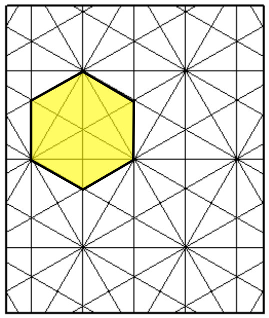
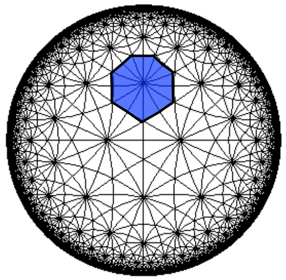
Right-angled pentagons



All surfaces can be built using one of 3 geometries

Klein
1870s

g=0 g=1 g=2,3,4,…

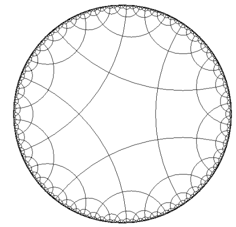




Uniformization

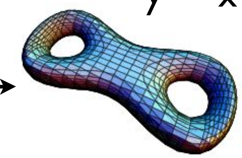
Hyperbolic surfaces = algebraic curves

Poincaré
1881

$\mathbb{H}$



G →



$y^2 = x^5 - 1$

To produce algebraic functions on $\mathbb{H}/G$: Poincaré series

$$\Theta(f) = \sum_{g \in G} g^*(f) \qquad f = f(z)\,dz^2$$

holomorphic quadratic differential on $\mathbb{H}$

The world of 3-manifolds

The Geometrization Conjecture

Thurston, 1982

All 3-manifolds can be built using just 8 geometries.

Perelman, 2003

The Eight Geometries

constant curvature	$\mathbb{S}^3$	$\mathbb{E}^3$	$\mathbb{H}^3$

$S^1 \to M \to \Sigma_g$

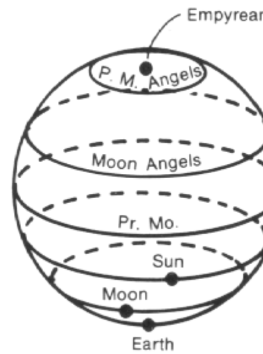
sphere	torus	g ≥ 2
$\mathbb{R} \times \mathbb{S}^2$	$\mathbb{E}^3$	$\mathbb{R} \times \mathbb{H}$
$\mathbb{S}^3$	Nil	$\widetilde{\mathrm{SL}_2\,\mathbb{R}}$

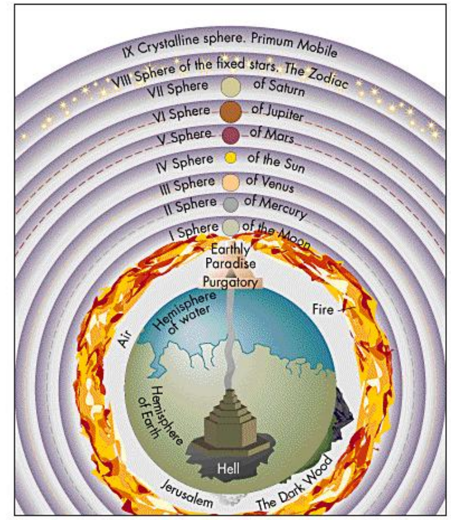
$S^1 x S^1 \to M \to S^1$

elliptic	parab	hyperbolic
$\mathbb{E}^3$	Nil	$\mathbb{R} \ltimes \mathbb{R}^2$

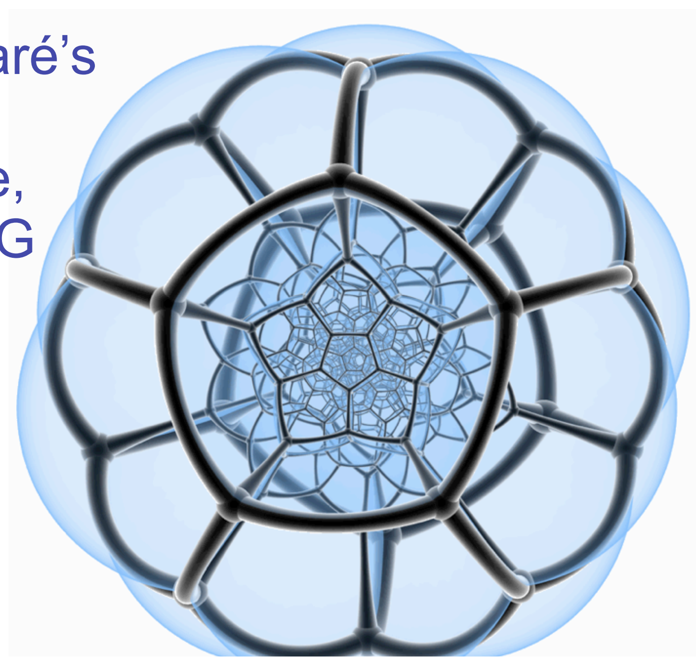
Products and twisted products: dimension 2+1/2

The 3-sphere

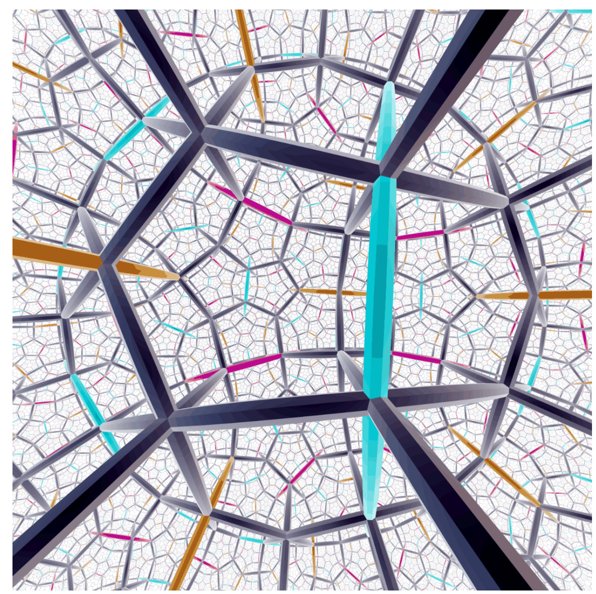




Poincaré's fake sphere, M=S³/G



Hyperbolic Geometry



Poincaré Conjecture

The only solution to $\pi_1(M^3)=(1)$ is $M^3=S^3$.

$\Uparrow$

Geometrization Conjecture

Every M^3 is built from geometric pieces; typically

$$\mathbb{H}^3/\Gamma \to M.$$

Fermat's Last Theorem

The only solutions to
$x^n+y^n = z^n$,
n>2, are trivial.

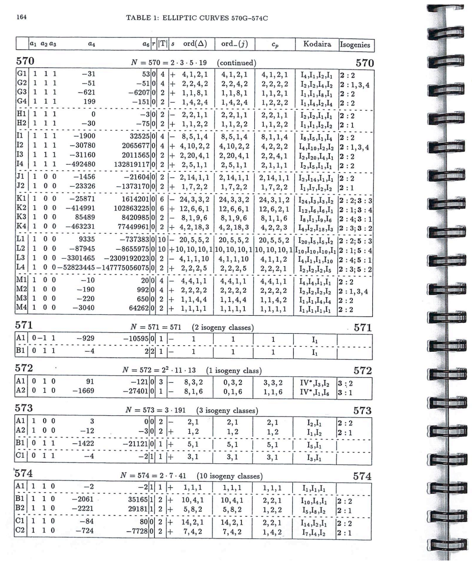
$\Uparrow$

Modularity Conjecture

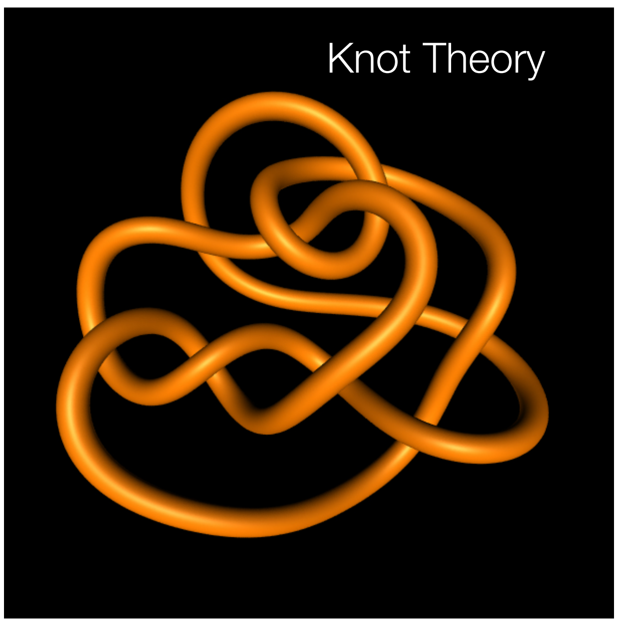
Every elliptic curve E over $\mathbb{Q}$ is dominated by a modular curve,

$$\mathbb{H}/\Gamma(N) \to E.$$

Catalog of elliptic curves/Q

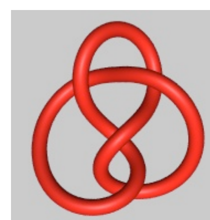


(Cremona, 1992)



Arithmetic Examples

$SL_2(Z[\omega])$ $SL_2(Z[i])$



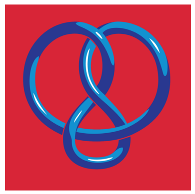
$$\mathbb{H}^3/\Gamma = S^3 \setminus K$$

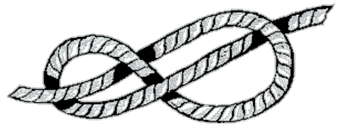
Mostow: Topology ⇒ Geometry

Hyperbolic volume

as a topological invariant

K =





$$\mathrm{vol}(S^3 - K) = 6Л(\pi/3) = 6\int_0^{\pi/3} \log \frac{1}{2\sin\theta}\, d\theta$$

$= 2.0298832128193...$

The Perko Pair

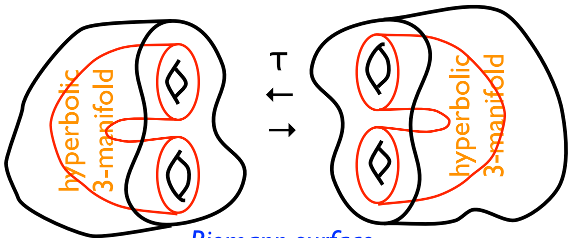
Hoste, Thistlethwaite and Weeks,1998:
The First 1,701,936 knots

(Up to 16 crossings)

Thurston's breakthroughs *1980s*

- *Almost all knot complements are hyperbolic.*
- *Almost all surgeries of S^3 along knot and links yield hyperbolic manifolds.*
- *The result of gluing together two hyperbolic 3-manifolds is hyperbolic, unless it contains a 2-torus.*

Evolution and gluing



Riemann surface boundary

Theorem(Thurston) M/τ hyperbolic ⇔

π_1(M/τ) does not contain $\mathbb{Z}^2$ ⇔

σ∘τ :Teich(∂M) → Teich(∂M) has a fixed point.

(Analytic approach: at a given X in Teich(∂M), we have |Dσ| ≤ $||\Theta_X||$; and $||\Theta_X|| < 1$.)

Evolution by curvature

Hamilton, Perelman

Ricci curvature flow on (M, g_t)

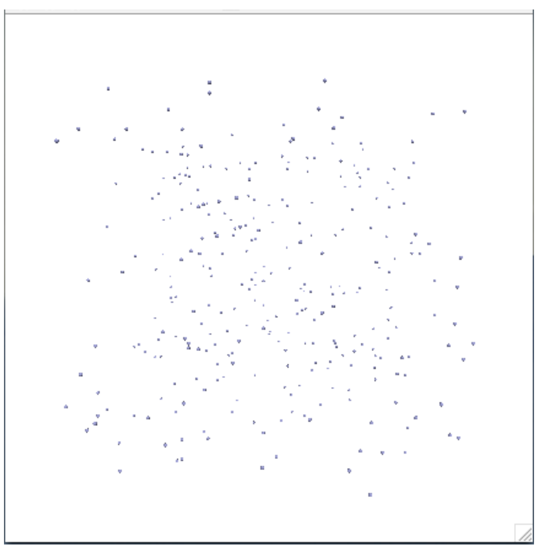
$$\frac{dg_{ij}}{dt} = -2R_{ij}$$

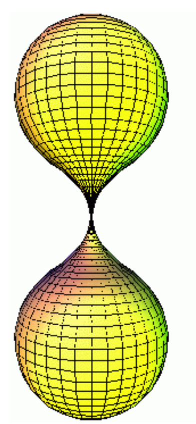
Darwin recognized that his weak and negative force... could only play [a] creative role if variation met three crucial requirements: copious in extent, small in range of departure from the mean, and isotropic.

Gould, 2002

Evolution by curvature

Singularities





Thanks to Dmitri Gekhtman

Perelman's papers

2002-3

- Singularities always undo connect sums
- Evolution with surgery continues for all time
- Eventually, architecture of M becomes visible

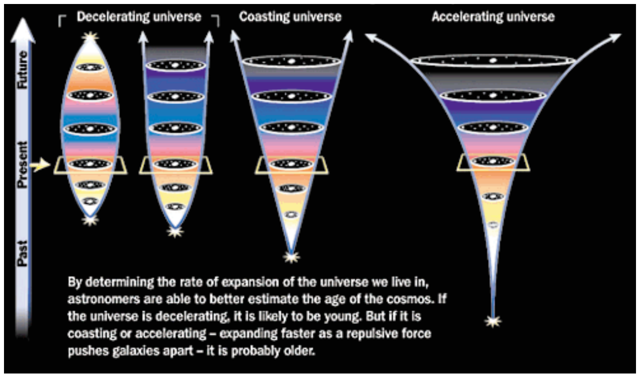
$\Longrightarrow$ Geometrization Conjecture is true

$\Longrightarrow$ Poincaré Conjecture is true

Cosmological Corollary

``General relativity places no constraints on the topology of the Universe.''

K>0 K=0 K<0



Iteration on Teich(∂M)	Ricci flow

Iteration on Teich(∂M)	Ricci flow
PL, geometric, classical, algorithmic	C^∞, infinite-dimensional, variable curvature

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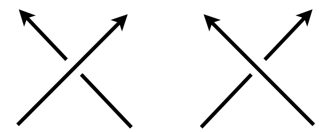
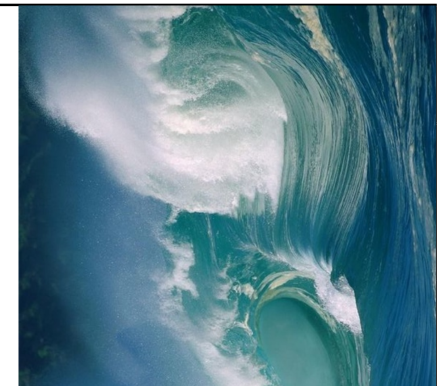
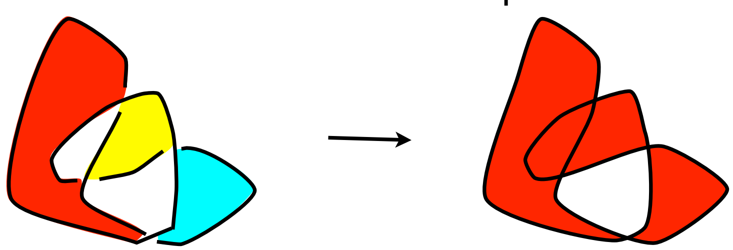
PL, geometric, classical, algorithmic	C^∞, infinite-dimensional, variable curvature
Bottom up Uses hierarchy	Top down General

Iteration on Teich(∂M)	Ricci flow
PL, geometric, classical, algorithmic	C^∞, infinite-dimensional, variable curvature
Bottom up Uses hierarchy	Top down General
Holomorphic Contracting	Monotone

Iteration on Teich(∂M)	Ricci flow
	
General case - surgery, cone manifolds?	Diffuse Welcomes singularities

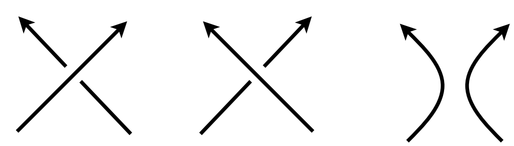
Surfaces in 3-manifolds?

Conjecture.
If M is a closed hyperbolic 3-manifold, then
a finite cover of M contains an incompressible surface.



Theorem (Kahn-Markovic).
If M is a closed, hyperbolic 3-manifold, then
$\pi_1(M)$ contains a surface group.

The Jones polynomial (1983)

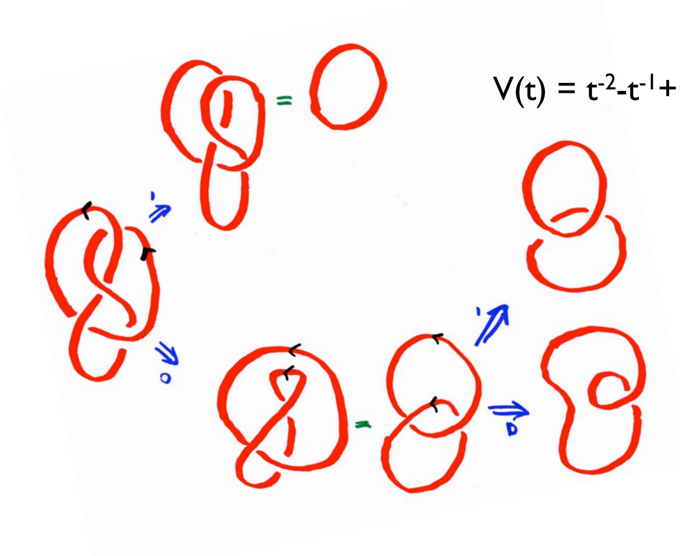


skein theory

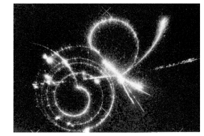
$$t^{-1}V_+ \;-\; tV_- \;=\; (t^{1/2} - t^{-1/2})\,V_0$$

$$V(O,t) = 1$$

Jones polynomial for figure 8 knot



$V(t) = t^{-2}-t^{-1}+1-t+t^{2}$

Quantum fields  (Witten)

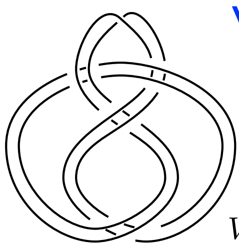
$$\langle K \rangle = \int \mathrm{Tr}(\oint_K A)\ e^{2\pi i k\, CS(A)}\ DA$$

$$= (q^{1/2}+q^{-1/2})\, V(K, 1/q)$$

$q = \exp(2\pi i/(2+k)) \to 1$ *as* $k \to \infty$

$\langle \text{unknot} \rangle \to 2$

Volume Conjecture



Cable K^2 for figure eight knot K

Murakami-Murakami
Kashaev

$$V_{n+1}(K,t) = \sum_{j=0}^{n/2} (-1)^j \binom{n-j}{j} V(K^{n-2j}, t)$$

$$\frac{2\pi \ \log |V_n(K, e^{2\pi i/n})|}{n} \to \text{hyperbolic vol}(S^3 - K)$$

quantum fields — *general relativity*