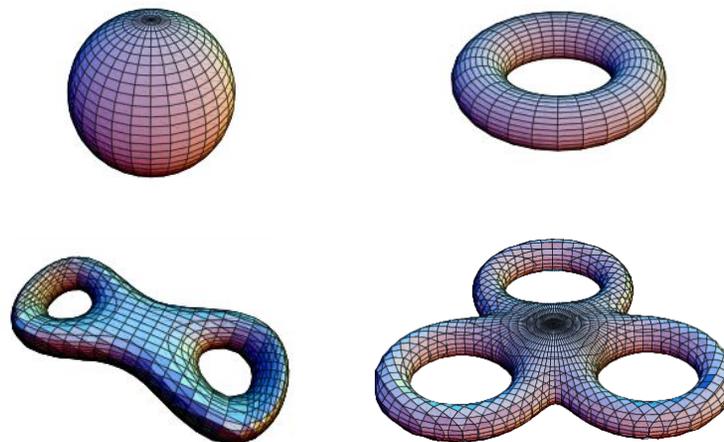


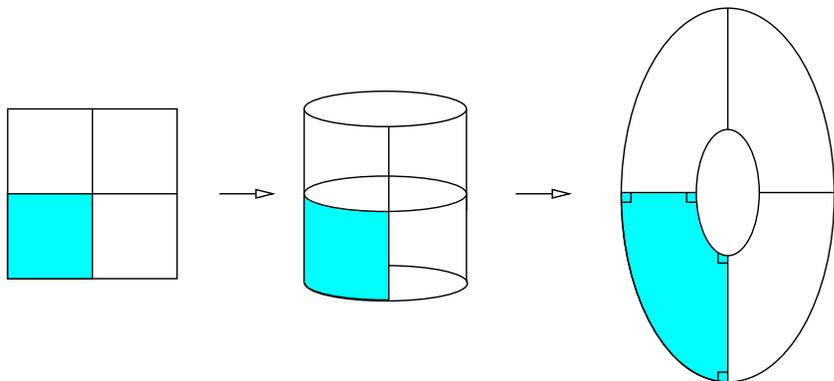
# *The Evolution of Geometric Structures on 3-Manifolds*

Curtis T McMullen  
Harvard University

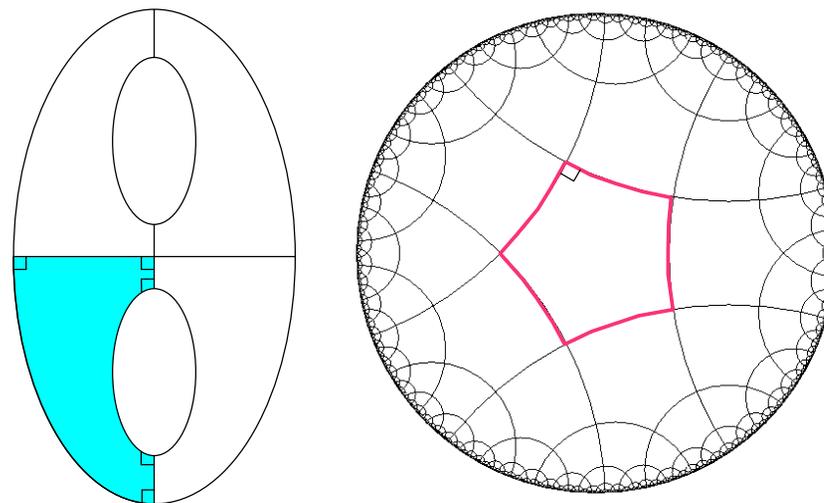
## Surfaces of genus 0, 1, 2, 3



## Squares tile the torus



## Right-angled pentagons





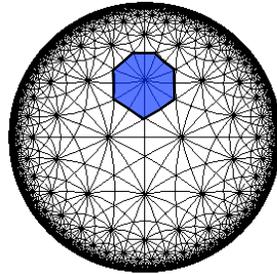
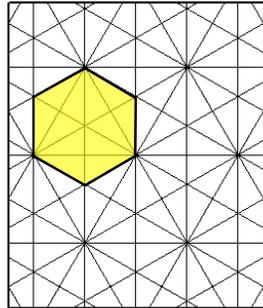
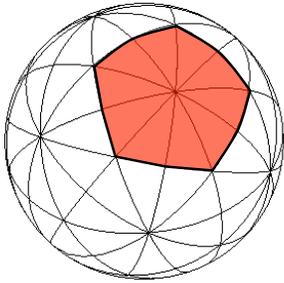
All surfaces can be built using one of 3 geometries

Klein  
1870s

$g=0$

$g=1$

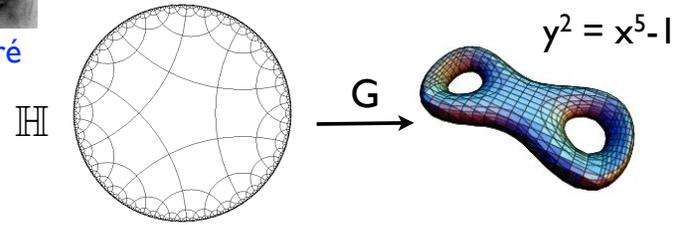
$g=2,3,4,\dots$



Poincaré  
1881

## Uniformization

Hyperbolic surfaces = algebraic curves

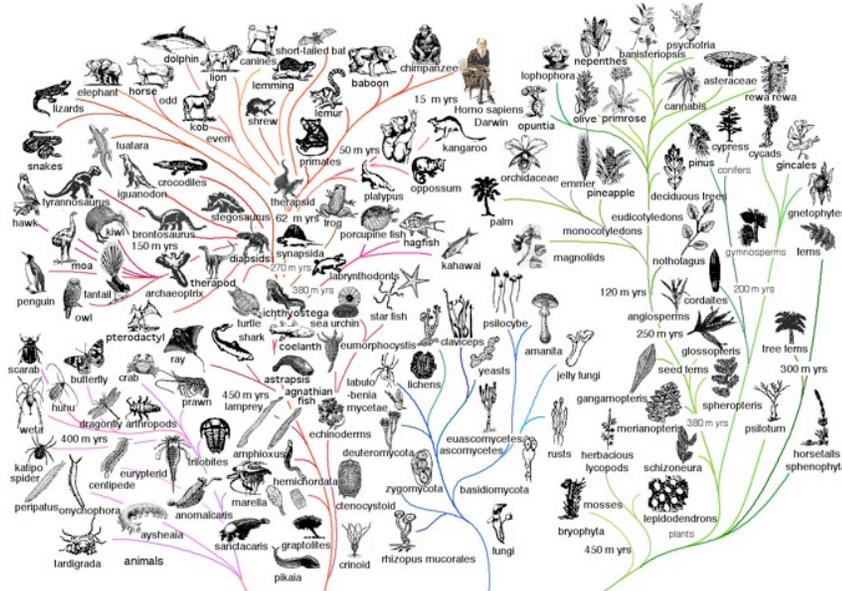


To produce algebraic functions on  $\mathbb{H}/G$ : Poincaré series

$$\Theta(f) = \sum_{g \in G} g^*(f) \quad f = f(z) dz^2$$

holomorphic quadratic differential on  $\mathbb{H}$

## The world of 3-manifolds



## The Geometrization Conjecture

Thurston, 1982

All 3-manifolds can be built using just 8 geometries.

Perelman, 2003

# The Eight Geometries

constant curvature

$S^3$	$E^3$	$H^3$
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$$S^1 \rightarrow M \rightarrow \Sigma_g$$

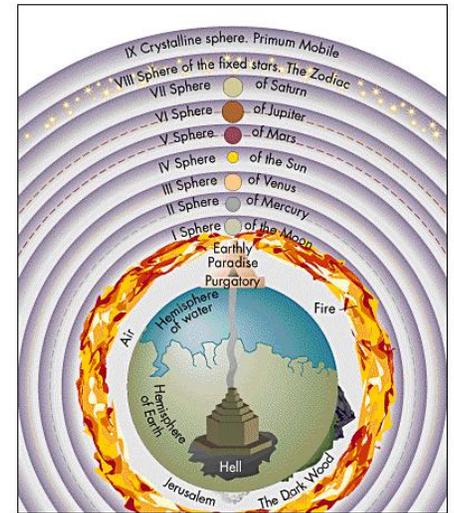
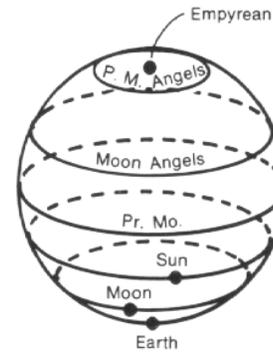
sphere	torus	$g \geq 2$
$\mathbb{R} \times S^2$	$E^3$	$\mathbb{R} \times \mathbb{H}$
$S^3$	<i>Nil</i>	$\widetilde{SL_2 \mathbb{R}}$

$$S^1 \times S^1 \rightarrow M \rightarrow S^1$$

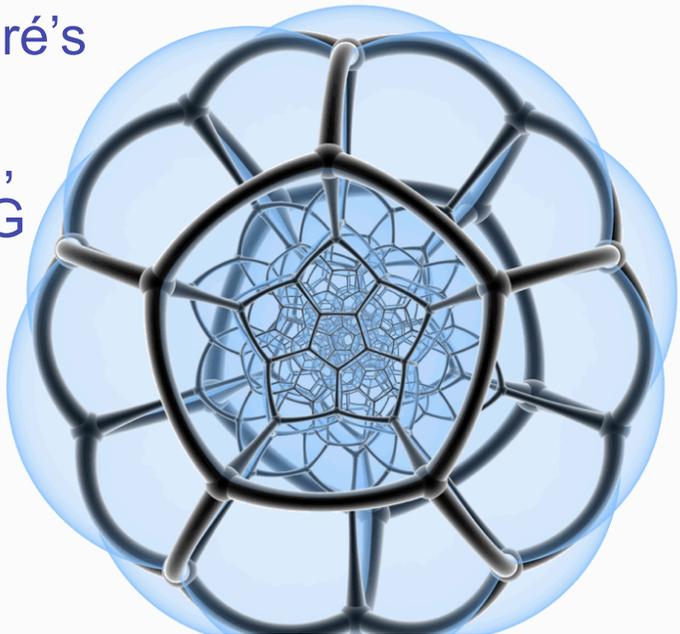
elliptic	parab	hyperbolic
$E^3$	<i>Nil</i>	$\mathbb{R} \times \mathbb{R}^2$

Products and twisted products: dimension 2+1/2

# The 3-sphere



Poincaré's fake sphere,  $M=S^3/G$



# Hyperbolic Geometry



## Poincaré Conjecture

The only solution to  $\pi_1(M^3)=(1)$  is  $M^3=S^3$ .



## Geometrization Conjecture

Every  $M^3$  is built from geometric pieces; typically

$$\mathbb{H}^3/\Gamma \rightarrow M.$$

## Fermat's Last Theorem

The only solutions to  $x^n+y^n = z^n$ ,  $n>2$ , are trivial.



## Modularity Conjecture

Every elliptic curve  $E$  over  $\mathbb{Q}$  is dominated by a modular curve,

$$\mathbb{H}/\Gamma(N) \rightarrow E.$$

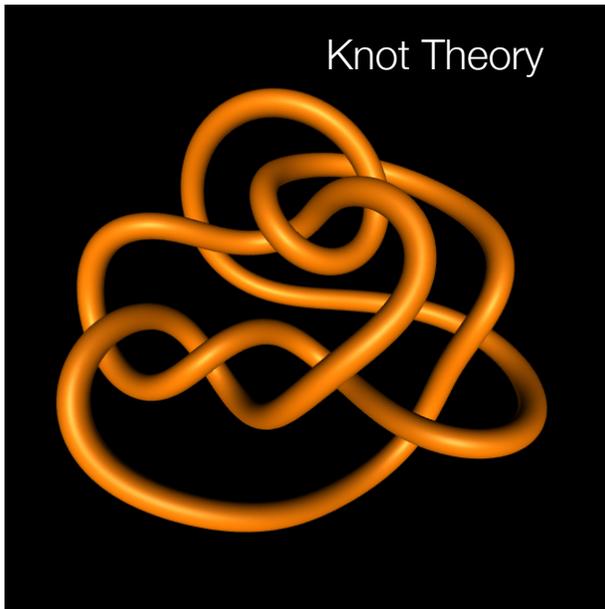
## Catalog of elliptic curves/ $\mathbb{Q}$

164 TABLE 1: ELLIPTIC CURVES 570G-574C

$[a_1, a_2, a_3]$	$a_4$	$a_5$	$ \Delta $	$\text{ord}(\Delta)$	$\text{ord}_2(\Delta)$	$c_4$	Kodaira	logarithm
<b>570</b> $N = 570 = 2 \cdot 3 \cdot 5 \cdot 19$ (continued)								
G1	1	1	-31	810	4	4,1,2,1	$I_4^*$	2:2
G2	1	1	-31	-510	4	2,2,4,2	$I_4^*$	2:1,3,4
G3	1	1	-31	-4070	2	1,1,6,1	$I_4^*$	2:2
G4	1	1	-319	-119	2	1,4,2,4	$I_4^*$	2:2
H1	1	1	0	-20	2	2,2,1,1	$I_4^*$	2:2
H2	1	1	-30	-750	2	1,1,2,2	$I_4^*$	2:1
I1	1	1	-1500	32550	4	4,1,2,1	$I_4^*$	2:2
J2	1	1	-30780	20656770	4	4,10,2,2	$I_4^*$	2:1,3,4
J3	1	1	-21150	2011650	2	2,20,4,1	$I_4^*$	2:2
J4	1	1	-492480	128191170	2	2,5,1,1	$I_4^*$	2:2
J1	1	0	-1456	-21640	2	2,14,1,1	$I_4^*$	2:2
K1	1	0	-23236	-1373700	2	1,7,2,2	$I_4^*$	3:1
K2	1	0	-25871	16142010	0	24,3,2,2	$I_4^*$	2:2,3,4
K3	2	0	-414991	102862250	0	12,6,6,1	$I_4^*$	2:1,3,4
K4	1	0	85489	42089610	2	4,1,6,6	$I_4^*$	4:1,1
K4	1	0	-65231	774499610	2	4,2,18,3	$I_4^*$	2:3,3,2
L1	1	0	9535	-7373810	10	20,5,5,2	$I_4^*$	2:2,3,4
L2	1	0	-87845	-8605070	10	10,10,10,10	$I_4^*$	2:1,4,4
L3	2	0	-3301465	-2309120230	2	4,1,1,10	$I_4^*$	2:4,6,1
L4	1	0	-33828445	-14775660750	2	4,2,2,5	$I_4^*$	2:3,4,2
M1	1	0	-10	100	4	4,4,1,1	$I_4^*$	2:2
M2	1	0	-190	920	4	2,2,2,2	$I_4^*$	2:1,3,4
M3	1	0	-220	6080	2	1,1,4,4	$I_4^*$	2:2
M4	1	0	-3040	642620	2	1,1,1,1	$I_4^*$	2:2
<b>571</b> $N = 571 = 571$ (2 logarithm classes)								
A1	0	-1	-929	-109550	1	1	$I_1$	
B1	0	1	-4	20	1	1	$I_1$	
<b>572</b> $N = 572 = 2^2 \cdot 11 \cdot 13$ (1 logarithm class)								
A1	1	0	81	-1210	3	3,3,2	$IV^*$	5:2
B1	0	1	-160	-275010	1	3,1,6	$IV^*$	3:1
<b>573</b> $N = 573 = 3 \cdot 191$ (2 logarithm classes)								
A1	1	0	3	0	2	2,1	$I_2^*$	2:2
A2	1	0	-12	-30	2	1,2	$I_2^*$	2:1
B1	0	1	-1432	-21210	1	3,1	$I_2^*$	
C1	0	1	-4	-20	1	3,1	$I_2^*$	
<b>574</b> $N = 574 = 2 \cdot 7 \cdot 41$ (10 logarithm classes)								
A1	1	0	-2	-20	1	1,1,1	$I_2^*$	
B1	1	0	-2061	351601	2	10,4,1	$I_2^*$	2:2
B2	1	0	-2231	391811	2	5,6,2	$I_2^*$	2:1
C1	1	0	-84	-800	2	14,2,1	$I_2^*$	2:2
C2	1	0	-724	-728	2	7,4,2	$I_2^*$	2:1

(Cremona, 1992)

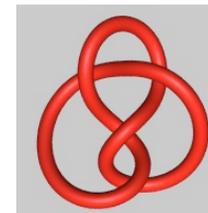
## Knot Theory



## Arithmetic Examples

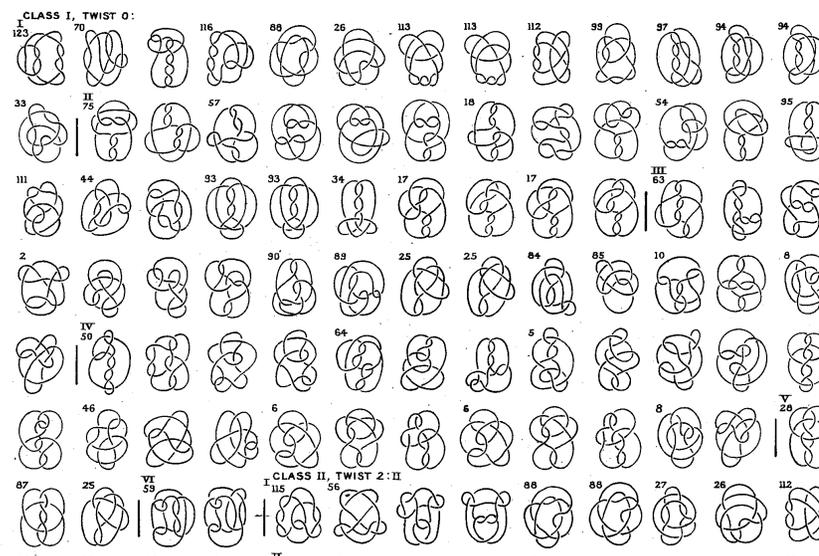
$$SL_2(\mathbb{Z}[\omega])$$

$$SL_2(\mathbb{Z}[i])$$



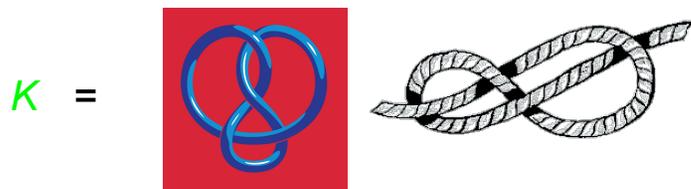
$$\mathbb{H}^3/\Gamma = S^3 \setminus K$$

Mostow: Topology  $\Rightarrow$  Geometry



# Hyperbolic volume

as a topological invariant



$$\text{vol}(S^3 - K) = 6\pi(\pi/3) = 6 \int_0^{\pi/3} \log \frac{1}{2 \sin \theta} d\theta$$

$$= 2.0298832128193\dots$$

## The Perko Pair



Hoste, Thistlethwaite and Weeks, 1998:  
The First 1,701,936 knots



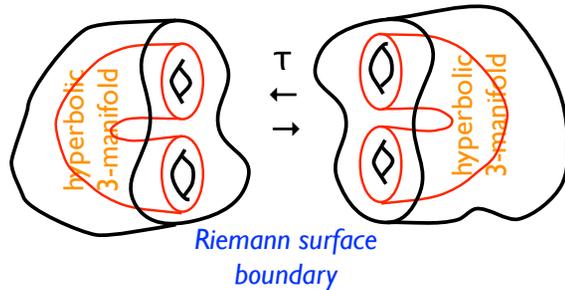
(Up to 16 crossings)

## Thurston's breakthroughs

1980s

- ✳ Almost all *knot complements* are hyperbolic.
- ✳ Almost all *surgeries* of  $S^3$  along knot and links yield hyperbolic manifolds.
- ✳ The result of *gluing* together two hyperbolic 3-manifolds is hyperbolic, unless it contains a 2-torus.

## Evolution and gluing



Theorem (Thurston)  $M/\Gamma$  hyperbolic  $\Leftrightarrow$

$\pi_1(M/\Gamma)$  does not contain  $\mathbb{Z}^2$   $\Leftrightarrow$

$\sigma \circ \tau : \text{Teich}(\partial M) \rightarrow \text{Teich}(\partial M)$  has a fixed point.

(Analytic approach: at a given  $X$  in  $\text{Teich}(\partial M)$ , we have  $|D\sigma| \leq \|\Theta_X\|$ ; and  $\|\Theta_X\| < 1$ .)

## Evolution by curvature

Hamilton, Perelman

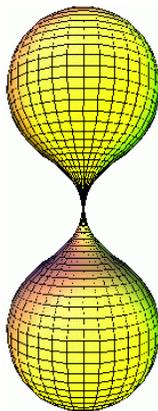
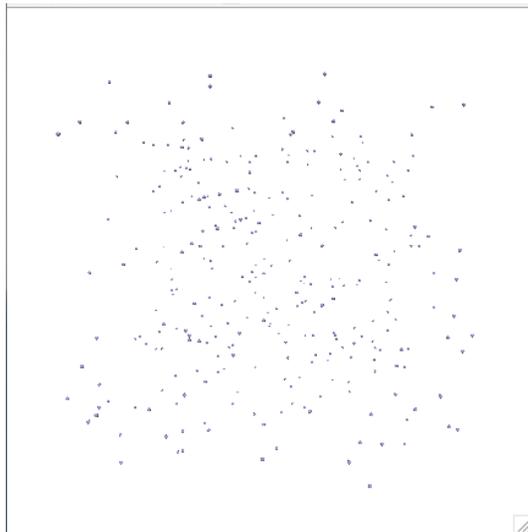
Ricci curvature flow on  $(M, g_t)$

$$\frac{dg_{ij}}{dt} = -2R_{ij}$$

Darwin recognized that his weak and negative force... could only play [a] creative role if variation met three crucial requirements: copious in extent, small in range of departure from the mean, and isotropic.

Gould, 2002

## Evolution by curvature Singularities



Thanks to  
Dmitri  
Gekhtman

## Perelman's papers

2002-3

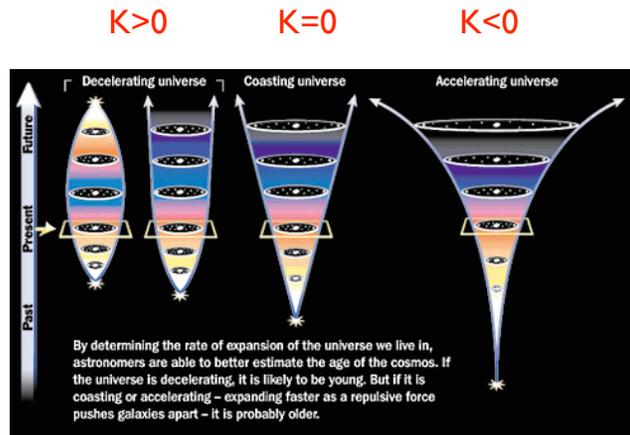
- \* Singularities always undo connect sums
- \* Evolution with surgery continues for all time
- \* Eventually, architecture of  $M$  becomes visible

$\Rightarrow$  Geometrization Conjecture is true

$\Rightarrow$  Poincaré Conjecture is true

## Cosmological Corollary

“General relativity places no constraints on the topology of the Universe.”



## Iteration on $\text{Teich}(\partial M)$

## Ricci flow


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## Ricci flow

PL, geometric, classical, algorithmic	$C^\infty$ , infinite-dimensional, variable curvature

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PL, geometric, classical, algorithmic	$C^\infty$ , infinite-dimensional, variable curvature
Bottom up Uses hierarchy	Top down General

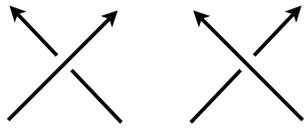
### Iteration on $\text{Teich}(\partial M)$

### Ricci flow

PL, geometric, classical, algorithmic	$C^\infty$ , infinite-dimensional, variable curvature
Bottom up Uses hierarchy	Top down General
Holomorphic Contracting	Monotone

### Iteration on $\text{Teich}(\partial M)$

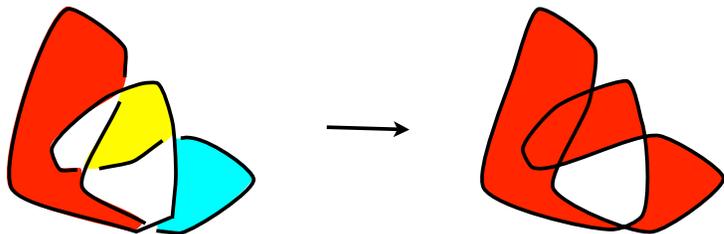
### Ricci flow

	
General case - surgery, cone manifolds?	Diffuse Welcomes singularities

## Surfaces in 3-manifolds?

*Conjecture.*

If  $M$  is a closed hyperbolic 3-manifold, then a finite cover of  $M$  contains an incompressible surface.



Theorem (Kahn-Markovic).

If  $M$  is a closed, hyperbolic 3-manifold, then  $\pi_1(M)$  contains a surface group.

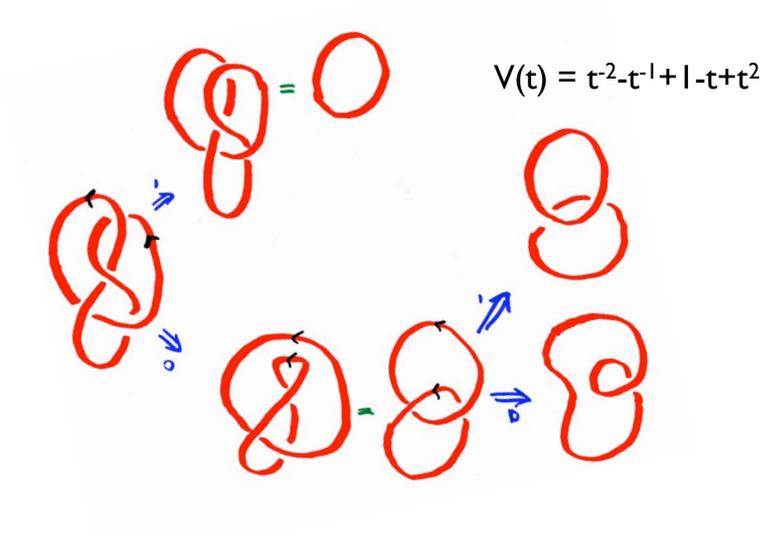
## The Jones polynomial (1983)

skein theory

$$t^{-1}V_+ - tV_- = (t^{1/2} - t^{-1/2})V_0$$

$$V(0, t) = 1$$

## Jones polynomial for figure 8 knot



## Quantum fields



(Witten)

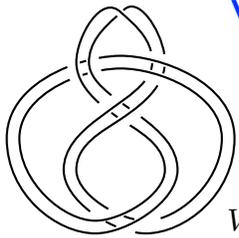
$$\langle K \rangle = \int \text{Tr}(\oint_K A) e^{2\pi i k \text{CS}(A)} \text{DA}$$

$$= (q^{1/2} + q^{-1/2}) V(K, 1/q)$$

$$q = \exp(2\pi i / (2+k)) \rightarrow 1 \text{ as } k \rightarrow \infty$$

$$\langle \text{unknot} \rangle \rightarrow 2$$

## Volume Conjecture



Murakami-Murakami  
Kashaev

$$V_{n+1}(K, t) = \sum_{j=0}^{n/2} (-1)^j \binom{n-j}{j} V(K^{n-2j}, t)$$

Cable  $K^2$  for figure  
eight knot  $K$

$$\frac{2\pi \log |V_n(K, e^{2\pi i/n})|}{n} \rightarrow \text{hyperbolic vol}(S^3 - K)$$

quantum fields

general relativity