The Gauss-Bonnet theorem for cone manifolds and volumes of moduli spaces

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Kappes–Möller (2012)
Thurston (1998)
Allendoerfer–Weil (1943)

Schwarz, Picard, Deligne–Mostow, Cohen–Wolfart, Parker, Sauter,....

Goals

- Calculate the
  complex hyperbolic volume
  of the moduli space $\mathcal{M}_{0,n}$.

- Method: compute the Euler characteristic of its completion, and apply
  Gauss-Bonnet for cone manifolds.

- Application: useful invariants of
  nonarithmetic subgroups of SU(1,n).

What is the Euler characteristic *of moduli space*?

$$\mathcal{M}_{0,n} = \{(b_1,\ldots,b_n) \in \widehat{\mathbb{C}}^n \ : \ b_i \neq b_j\} / \operatorname{Aut} \widehat{\mathbb{C}}$$

manifold

$$\begin{array}{ccc} \Sigma_{0,n-1} & \rightarrow & \mathcal{M}_{0,n} \\ & & \downarrow \\ & & \mathcal{M}_{0,n-1} \end{array}$$

Fibration

$\chi(\Sigma_{0,n-1}) = -(n-3)$

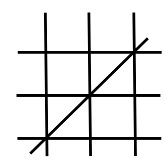
$$\chi(\mathcal{M}_{0,n}) = (-1)^{n+1}(n-3)!$$

Compactified moduli space?

n	3	4	5	6	7
$\chi(\mathcal{M}_{0,n})$	1	-1	2	-6	24
$\chi(\overline{\mathcal{M}_{0,n}})$	1	2	7	34	213

Deligne - Mumford

Example: n=5



$$\chi(\mathbb{P}^1 \times \mathbb{P}^1 - 7\mathbb{P}^1 + 12\mathbb{P}^0) = 4 - 14 + 12 = 2$$

$$\chi(\mathbb{P}^1 \times \mathbb{P}^1 + 3\mathbb{P}^1 - 3\mathbb{P}^0) = 4 + 6 - 3 = 7$$

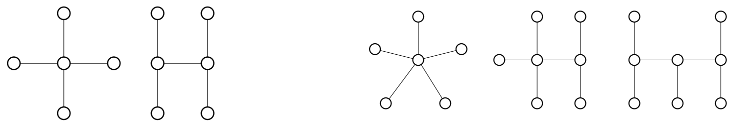
Generating functions

Corollary 2 (Getzler) *The generating functions*

$$f(x) = x - \sum_{n=2}^{\infty} \chi(\mathcal{M}_{0,n+1}) \frac{x^n}{n!} \quad \textit{and} \quad g(x) = x + \sum_{n=2}^{\infty} \chi(\overline{\mathcal{M}}_{0,n+1}) \frac{x^n}{n!}$$

are formal inverses of one another.

Universal; via stable trees (M, L'Ens. math.)



n=4 n=5

Moduli space is totally inhomogeneous

Theorem (Royden).
The automorphism group of the universal cover

$$\mathcal{T}_{0,n} \to \mathcal{M}_{0,n} \qquad \text{(n>4)}$$

as a complex manifold, is discrete.

In particular, $\mathcal{T}_{0,n}$ looks nothing like

$$\mathbb{CH}^{n-3} \cong B^{n-3} \subset \mathbb{C}^{n-3}$$

Cor: $\mathcal{M}_{0,n} \neq \mathbb{CH}^{n-3}/\Gamma$ for n>4.

Moduli spaces of polyhedra...

Fix $\mu_1, \ldots, \mu_n, 0 < \mu_i < 1, \sum \mu_i = 2$.

Any (b_i) determines a meromorphic 1-form on $\widehat{\mathbb{C}}$:

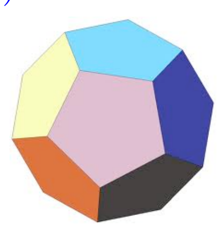
$$\omega = \frac{dx}{\prod_1^n (x - b_i)^{\mu_i}} \qquad (\omega) = -\sum \mu_i b_i$$

divisor of degree -2

$(\widehat{\mathbb{C}}, |\omega|) \cong$ convex polyhedron in $\mathbb{R}^3$

Cone angles $2\pi(1 - \mu_i)$.

Example: n=20, μ_i = 1/10



... are complex hyperbolic after all

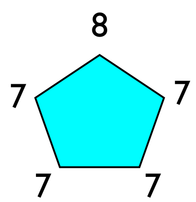
$\mathcal{M}_{0,n}(\mu)$ = moduli space of cone metrics on S^2 with given angles

Theorem: $\mathcal{M}_{0,n}(\mu)$ is naturally a complex hyperbolic manifold

(locally $\mathbb{CH}^{n-3}$, via periods of ω)

Schwarz, Picard, Deligne-Mostow, Thurston, 1986 Math Olympiad

Example: μ = (7,7,7,7,8)/18

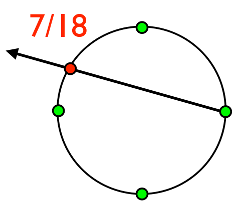


$$\mathcal{M}_{0,5} \to \mathcal{M}_g,\ g = 25$$

$$X : y^{18} = (x - b_1)\cdots(x - b_4)$$

$\mathbb{Z}/18$ acts on $H^1(X)$

$[\omega] = [dx/y^7] \in H^1(X)_q \qquad q = \zeta_{18}^{-7}$



Signature (1,2)

$(b_i) \mapsto$ (positive line in $\mathbb{C}^{1,2}$) $\cong \mathbb{CH}^2$

Also get rep of braid group $B_4 \to U(1,2)$

(Burau)

What is the volume of moduli space?

Theorem 1.2 *The complex hyperbolic volume of moduli space satisfies*

$$\mathrm{vol}(\mathcal{M}_{0,n}, g_\mu) = C_{n-3} \sum_{\mathcal{P}} (-1)^{|\mathcal{P}|+1}(|\mathcal{P}| - 3)! \prod_{B\in\mathcal{P}} \max\left(0, 1 - \sum_{i\in B} \mu_i\right)^{|B|-1}.$$

$\mathcal{P}$: partitions of $\{1, \ldots, n\}$ into blocks B.

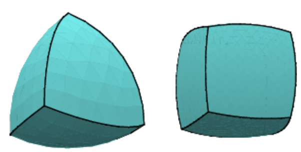
special cases: Parker, Sauter

General approach: GB + Euler characteristic

Thurston:

The metric completion $\overline{\mathcal{M}_{0,n}}(\mu)$ is a $\mathbb{CH}^{n-3}$ cone manifold.

Cone manifolds



Example: *Glue together spherical polyhedra along congruent faces in pairs.*

Gauss-Bonnet (M):

A compact cone–manifold of dimension n satisfies

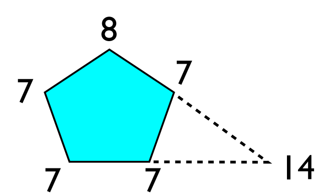
$$\int_{M[n]} \Psi(x)\, dv(x) = \sum_{\sigma} \chi(M^\sigma)\Theta^\sigma.$$

= Sum $_{\text{strata}}$ (Euler char) x (Solid normal angle)

Proof of volume formula for $\mathcal{M}_{0,n}(\mu)$

$$C \cdot \text{Volume} \overset{GB}{=} \sum_{\sigma} \chi(M^\sigma)\Theta^\sigma$$

$$= \sum_{\mathcal{P}} (-1)^{|\mathcal{P}|+1}(|\mathcal{P}| - 3)! \prod_{B\in\mathcal{P}} \max\left(0, 1 - \sum_{i\in B} \mu_i\right)^{|B|-1}.$$



$\mathcal{P}$ = (7+7,7,7,8) contributes a stratum $\simeq \mathcal{M}_{0,4}$ with Θ = (1-14/18) = 2/9.

no stratum unless $\sum_B \mu_i < 1$.

q	(p_i)	$\chi(P(\mu))$	$\chi(M(\mu))$
3	1 1 1 1 1 1	-4/9	-1/1620
3	2 1 1 1 1	1/3	1/72
4	1 1 1 1 1 1 1 1	-15/64	-1/172032
4	2 1 1 1 1 1 1	25/128	5/18432
4	3 1 1 1 1 1	-1/16	-1/1920
4	2 2 1 1 1 1	-1/4	-1/192
4	3 2 1 1 1	3/16	1/32
4	2 2 2 1 1	3/8	1/32
5	2 2 2 2 2	3/5	1/200
6	1 1 1 1 1 1 1 1 1 1 1 1	-28315/419904	-809/5746705367040
6	2 1 1 1 1 1 1 1 1 1 1	5663/93312	809/48372940800
6	3 1 1 1 1 1 1 1 1 1	-119/3888	-17/201553920
6	2 2 1 1 1 1 1 1 1 1	-287/4374	-41/50388480
6	4 1 1 1 1 1 1 1 1	49/5832	7/33592320
6	3 2 1 1 1 1 1 1 1	2107/46656	301/33592320
6	5 1 1 1 1 1 1 1	-1/1296	-1/6531840
6	2 2 2 1 1 1 1 1 1	637/7776	637/33592320
6	4 2 1 1 1 1 1 1	-13/648	-13/466560
6	3 3 1 1 1 1 1 1	-11/216	-11/311040
6	3 2 2 1 1 1 1 1	-91/1296	-91/311040
6	5 2 1 1 1 1 1	5/1296	1/31104
6	4 3 1 1 1 1 1	55/1296	11/31104
6	2 2 2 2 1 1 1 1	-13/108	-13/62208
6	4 2 2 1 1 1 1	5/108	5/5184
6	3 3 2 1 1 1 1	55/648	55/31104
6	5 3 1 1 1 1	-1/54	-1/1296
6	4 4 1 1 1 1	-2/27	-1/648
6	3 2 2 2 1 1 1	55/432	55/15552
6	5 2 2 1 1 1	-1/54	-1/648
6	4 3 2 1 1 1	-5/54	-5/324
6	3 3 3 1 1 1	-1/9	-1/324
6	5 4 1 1 1	1/12	1/72
6	2 2 2 2 2 1 1	5/24	1/1152
6	4 2 2 2 1 1	-1/9	-1/108
6	3 3 2 2 1 1	-5/27	-5/216
6	5 3 2 1 1	1/12	1/24
6	4 4 2 1 1	1/6	1/24
6	4 3 3 1 1	1/6	1/24
6	3 2 2 2 2 1	-5/18	-5/432
6	5 2 2 2 1	1/12	1/72
6	4 3 2 2 1	1/4	1/8
6	3 3 3 2 1	1/3	1/18
6	3 3 2 2 2	1/2	1/24
8	3 3 3 3 3 1	-33/128	-11/5120
8	6 3 3 3 1	9/64	3/128
8	5 5 2 2 2	9/32	3/128
8	4 3 3 3 3	9/16	3/128

Table 3. Euler characteristics of the 94 orbifolds $M(\mu)$ and their cone manifold covers $P(\mu)$, with $(\mu_i) = (p_i/q)$.

94 orbifolds of Deligne and Mostow

q	(p_i)	$\chi(P(\mu))$	$\chi(M(\mu))$
9	4 4 4 4 2	13/27	13/648
10	7 4 4 4 1	3/20	1/40
10	3 3 3 3 3 3 2	293/1000	293/720000
10	6 3 3 3 3 2	-26/125	-13/1500
10	9 3 3 3 2	3/100	1/200
10	6 6 3 3 2	3/10	3/40
10	5 3 3 3 3 3	-17/50	-17/6000
10	8 3 3 3 3	3/25	1/200
10	6 5 3 3 3	39/100	13/200
12	8 5 5 5 1	7/48	7/288
12	7 7 2 2 2 2 2	575/10368	115/497664
12	9 7 2 2 2 2	-23/432	-23/10368
12	7 7 4 2 2 2	-23/216	-23/2592
12	11 7 2 2 2	1/48	1/288
12	9 9 2 2 2	1/8	1/96
12	9 7 4 2 2	7/48	7/96
12	7 7 6 2 2	1/6	1/24
12	7 7 4 4 2	7/24	7/96
12	7 5 3 3 3 3	-31/144	-31/3456
12	5 5 5 3 3 3	-23/72	-23/2592
12	10 5 3 3 3	1/12	1/72
12	8 7 3 3 3	13/48	13/288
12	8 5 5 3 3	7/24	7/96
12	7 6 5 3 3	17/48	17/96
12	6 5 5 5 3	1/2	1/12
12	7 5 4 4 4	11/24	11/144
12	6 5 5 4 4	13/24	13/96
12	5 5 5 5 4	7/12	7/288
14	11 5 5 5 2	6/49	1/49
14	8 5 5 5 5	24/49	1/49
15	8 6 6 6 4	37/75	37/450
18	11 8 8 8 1	13/108	13/648
18	13 7 7 7 2	4/27	2/81
18	10 10 7 7 2	13/54	13/216
18	14 13 3 3 3	13/108	13/648
18	10 7 7 7 5	13/27	13/162
18	8 7 7 7 7	16/27	2/81
20	14 11 5 5 5	99/400	33/800
20	13 9 6 6 6	69/200	23/400
20	10 9 9 6 6	99/200	99/800
24	19 17 4 4 4	11/96	11/576
24	14 9 9 9 7	11/24	11/144
30	26 19 5 5 5	4/75	2/225
30	23 22 5 5 5	37/300	37/1800
30	22 11 9 9 9	16/75	8/225
42	34 29 7 7 7	61/588	61/3528
42	26 15 15 15 13	61/147	61/882

Table 3. (continued)

Proof of cone GB uses polyhedral GB...

1943

Theorem 2.1 (Allendoerfer–Weil) *The Euler characteristic of a compact Riemannian polyhedron M of dimension n satisfies*

$$(-1)^n \chi'(M) = \int_{M[n]} \Psi(x)\, dv(x) + \sum_{r=0}^{n-1} \int_{M[r]} dv(x) \int_{N(x)^*} \Psi(x,\xi)\, d\xi.$$

outer angles

$$\Psi(x) = \frac{2}{\omega_n} \cdot \frac{1}{2^{n/2} n!} \sum_{i,j \in S_n} \frac{\epsilon(i)\epsilon(j)}{g} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{n-1} i_n j_{n-1} j_n}.$$

intrinsic K

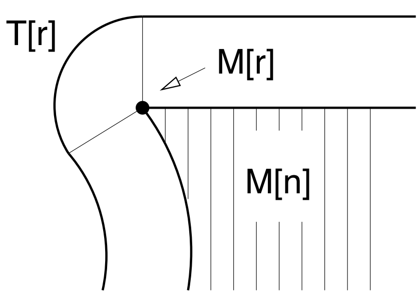
$$\Psi(x,\xi) = \sum_{0 \le 2f \le r} \Psi_{r,f}(x,\xi), \quad \text{where}$$

$$\Psi_{r,f}(x,\xi) = \frac{2}{\omega_{2f}\omega_{n-2f-1}} \cdot \frac{1}{2^f (2f)! (r-2f)!} \cdot \sum_{i,j \in S_r} \frac{\epsilon(i)\epsilon(j)}{\gamma} \times R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2f-1} i_{2f} j_{2f-1} j_{2f}} \Lambda_{i_{2f+1} j_{2f+1}}(\xi) \cdots \Lambda_{i_r j_r}(\xi).$$

Hopf / AW / Chern

extrinsic K

...which in turns comes from Weyl's tube formula. *1939*



Challenges in proving GB:

1: Inner angles versus outer angles

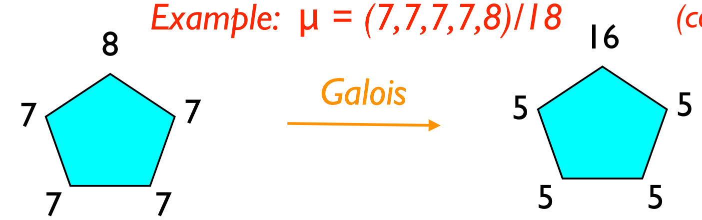
2: Complex hyperbolic case:

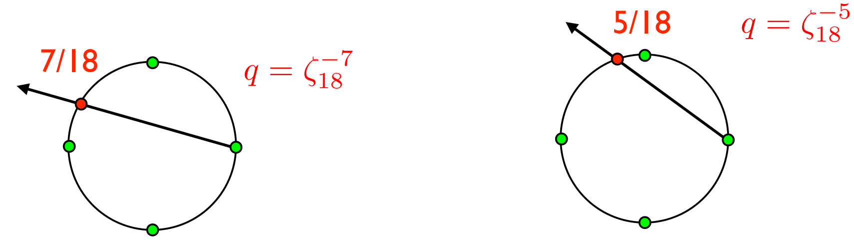
No odd-dimensional totally geodesic submanifolds.
Must fracture and reassemble.

Non arithmetic groups in SU(1,2)

Deligne-Mostow

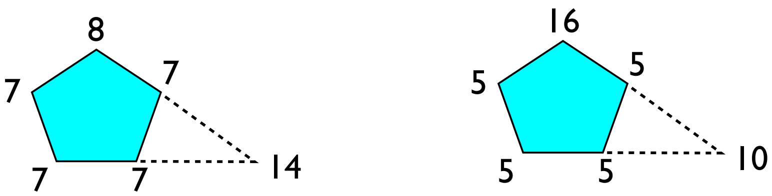
Example: μ = (7,7,7,7,8)/18 *(continued)*





Signature (1,2) *Signature (1,2)*

Geometry of nonarithmetic lattices



(18-14)/36 = 1/9

(18-10)/36 = 2/9

$\overline{\mathcal{M}_{0,n}}(\mu) \leftarrow \mathcal{M}_{0,n} \rightarrow \overline{\mathcal{M}_{0,n}}(\mu')$

$\overline{\mathcal{M}_{0,n}}(\mu) \xrightarrow{\text{holomorphic}} \overline{\mathcal{M}_{0,n}}(\mu')$

orbifold $\cong \mathbb{CH}^2/\Gamma$

cone manifold, holonomy Γ'

$(\Gamma \text{ discrete}) \cong (\Gamma' \text{ dense})$ in $U(1,2)$.

New Invariants

Volume ratios: $\rho(\mu,\mu') = \dfrac{\mathrm{vol}(\mathcal{M}_{0,n}(\mu'))}{\mathrm{vol}(\mathcal{M}_{0,n}(\mu))}$

Kappes-Möller

These are the same for all subgroups of finite index in Γ.

Cor. The 16 nonarithmetic lattices arising from moduli spaces fall into 10 commensurability classes.

16 Nonarithmetic Lattices in SU(1,2)

9 Volume invariants

10 Commensurability Classes

* different trace field

q	(p_i)	$\{\rho(\mu,\nu)\}$
12	7 5 3 3 3 3	1/93
12	8 7 3 3 3	1/13
12	6 5 5 4 4	1/13
12	7 6 5 3 3	1/17
18	13 7 7 7 2	1/16
18	8 7 7 7 7	1/16
20	14 11 5 5 5	1/33, 4/33
20	10 9 9 6 6	1/33, 4/33
20	13 9 6 6 6	1/46
12	7 5 4 4 4	1/22 *
24	19 17 4 4 4	1/22
24	14 9 9 9 7	1/22
15	8 6 6 6 4	1/37, 4/37
30	23 22 5 5 5	1/37, 4/37
42	34 29 7 7 7	1/61, 4/61
42	26 15 15 15 13	1/61, 4/61

Lyapunov exponent 1/3

Coda: nonarithmetic Fuchsian groups

μ = (3,3,7,7)/10

μ′ = (1,1,9,9)/10