Riemann Surfaces in

Dynamics, Topology and Arithmetic

Curtis T. McMullen

Harvard University, Cambridge, MA

Topics

I. The hyperbolic Laplacian and the Mandelbrot set

II. The shape of moduli space

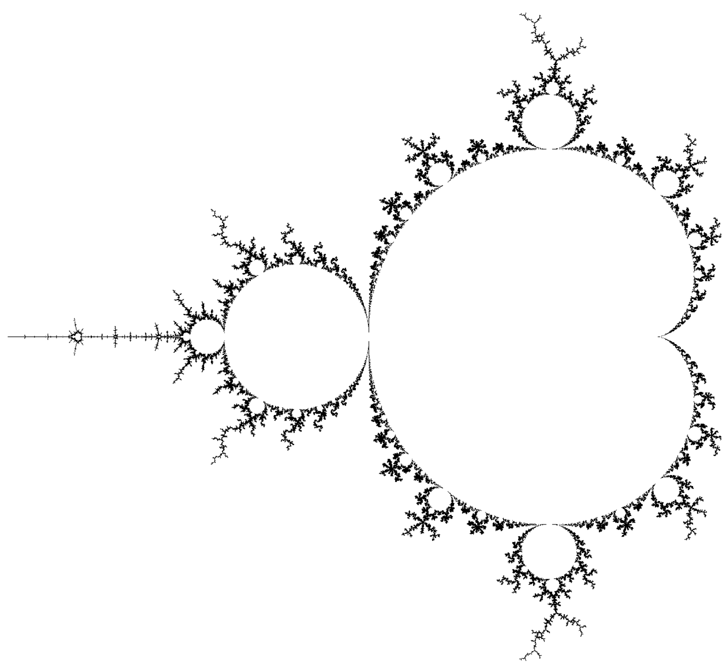
III. From dynamics on surfaces to rational points on curves

The Mandelbrot set

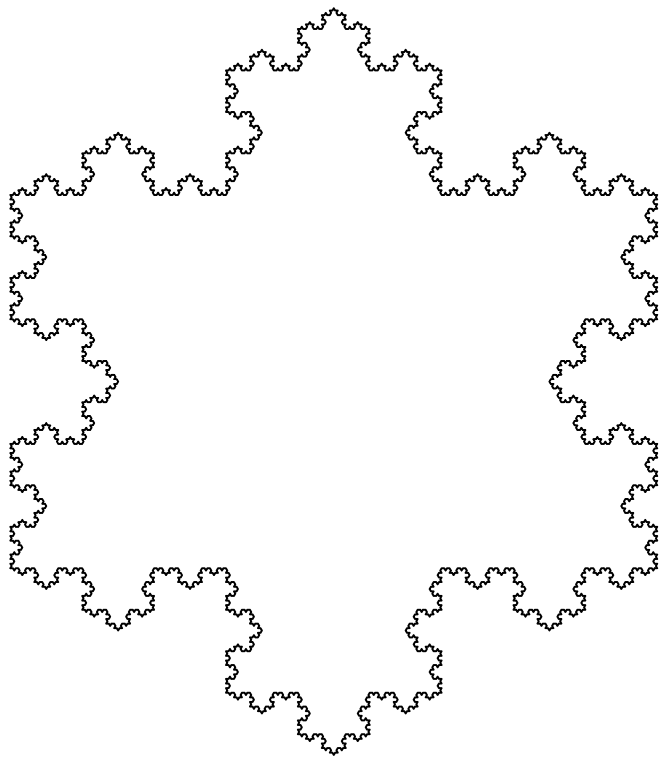
$$\begin{aligned} f_c(z) &= z^2 + c \\ M &= \{c \;:\; f_c^n(0) \not\to \infty\} \end{aligned}$$

Theorem *∂M has Hausdorff dimension two.*

—Shishikura, 1994.

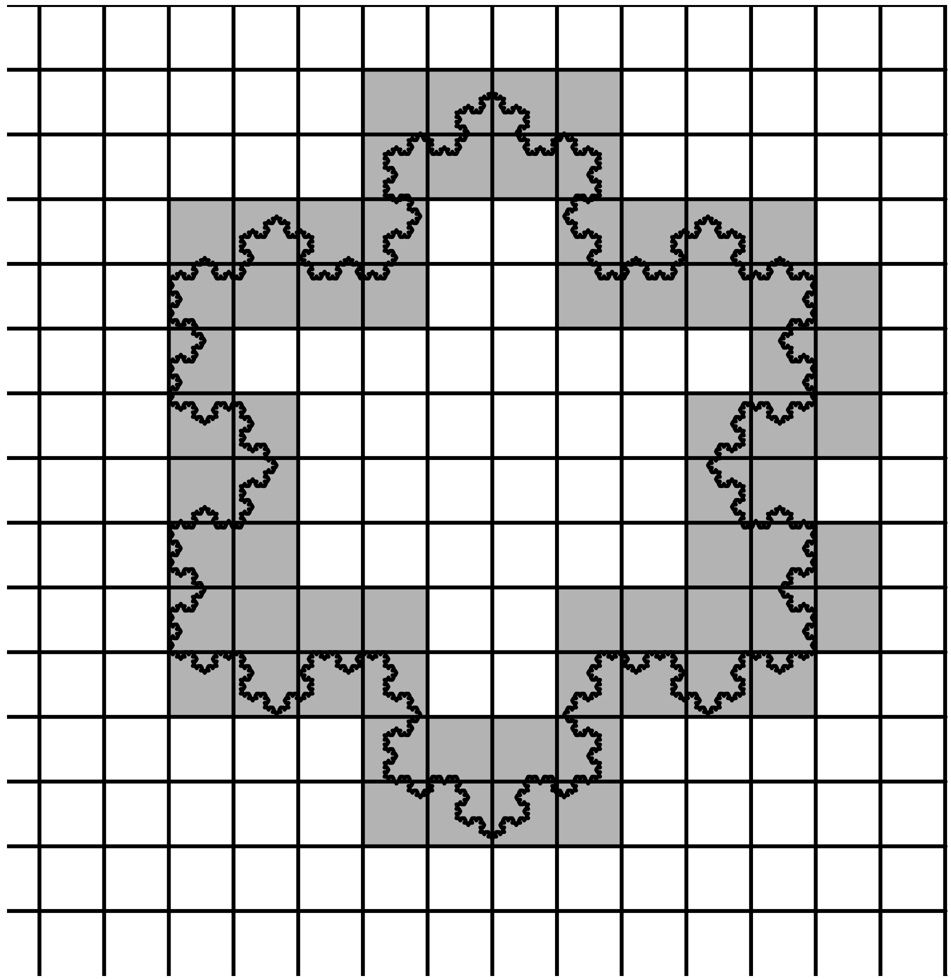


The dimension of a snowflake



$$\begin{aligned} S &= 4 \text{ copies of } \frac{1}{3}S \quad \Longrightarrow \\ 1 &= 4(1/3)^D \quad \Longrightarrow \\ \dim(S) &= \frac{\log 4}{\log 3} = 1.26186\ldots \end{aligned}$$

Box counting



$$(\text{Number of boxes of side } r) \sim \left(\frac{1}{r}\right)^D$$

$$D = \text{dimension}$$

Motions of hyperbolic space $\mathbb{H}^{n+1}$

Hyperbolic manifolds:

$$M = \mathbb{H}^{n+1}/\Gamma,$$

$\Gamma \subset \mathrm{Isom}(\mathbb{H}^{n+1})$ a discrete group of motions.

Sphere at infinity:

$$S^n_\infty \quad = \quad \mathbb{R}^n \cup \{\infty\} \quad = \quad \partial\mathbb{H}^{n+1}.$$

Hyperbolic motions:

$$(\text{Isometries of } \mathbb{H}^{n+1}) \iff (\text{Conformal automorphisms of } S^n_\infty = \partial\mathbb{H}^{n+1})$$
$$(\text{Isometries of } \mathbb{H}^3) \iff (\text{Automorphisms of } S^2_\infty \cong \widehat{\mathbb{C}}) \iff$$
$$\left(\text{Möbius transformations} \quad z \mapsto \frac{az+b}{cz+d}\right)$$

Limit set:

$$\Lambda = \{\text{limit points of } \Gamma \cdot z\} \subset S^n_\infty,$$

any $z \in S^n_\infty$. (Locus of chaotic dynamics)

The hyperbolic Laplacian

The least eigenvalue of the Laplacian:

$$\begin{aligned}\lambda_0(X) &= \inf\{\lambda > 0 \ : \ \Delta f = \lambda f, f \in L^2(X)\} \\ &= \inf_{f \in C_0^\infty(X)} \int_X |df|^2 \Big/ \int_X |f|^2\end{aligned}$$

The dimension of the limit set:

$$D = \dim(\Lambda).$$

Theorem (Cheeger).

$$\lambda_0(X) \text{ is small} \iff$$
$$|\partial S|/|S| \text{ is small, for some compact } S \subset X.$$

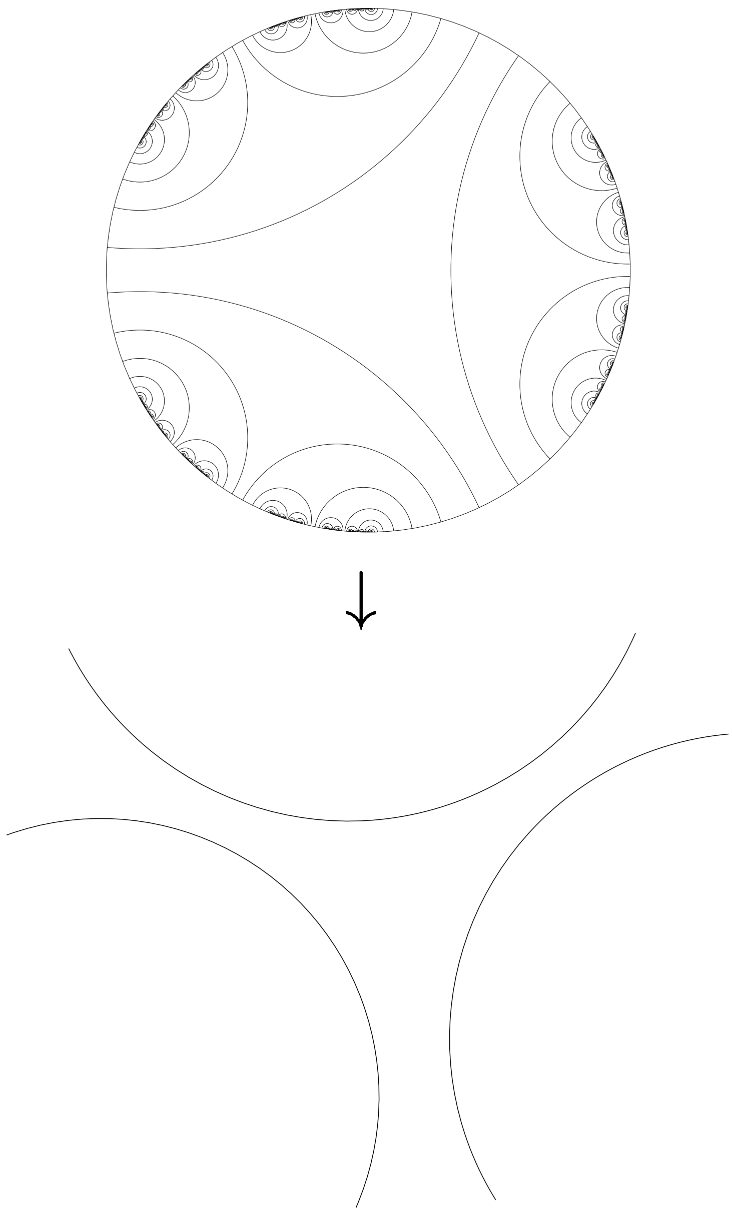
Theorem (Patterson).
The dimension and the Laplacian are related by

$$\lambda_0 = D(n - D)$$

when $\Lambda \subset S^n_\infty$.

Small eigenvalue and large limit set

The limit set is a Cantor set with $D \approx 0.70055063\ldots$



Theorem. $\dim \Lambda \to 1$ as $|\partial S| \to 0$.

Cusps

Γ has a **cusp** if (up to conjugacy)

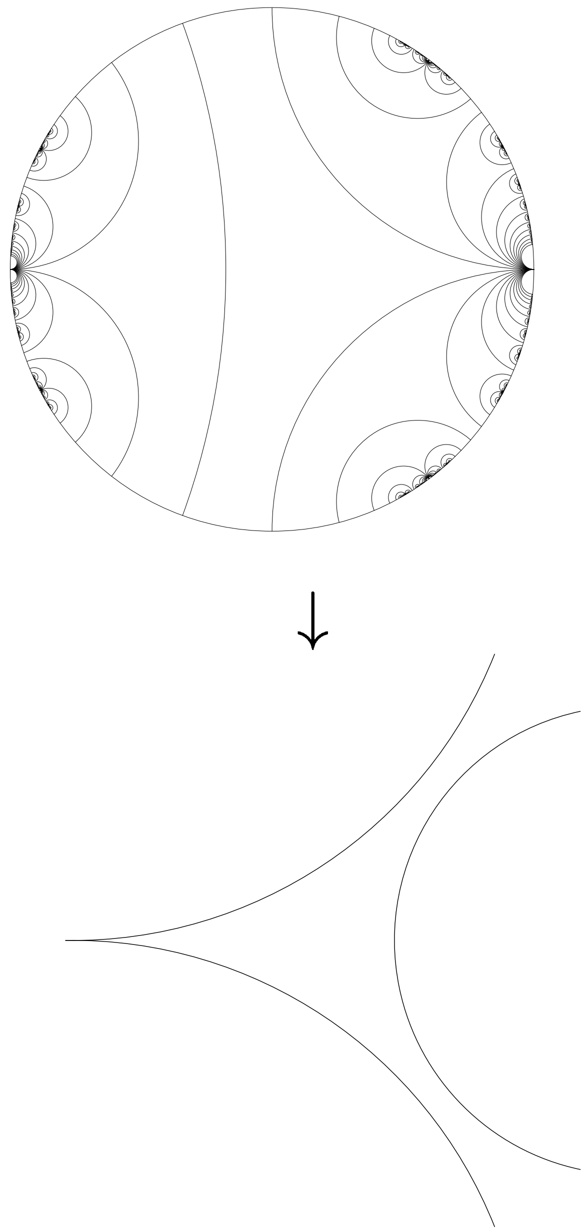
$$\Gamma \supset \{z \mapsto z + \ell, \ell \in L\}$$

for some discrete subgroup

$$L \subset \mathbb{R}^n.$$

The cusp has **rank** r if $L \cong \mathbb{Z}^r$.

Surface with a rank one cusp



Cusps and dimension

Theorem. Γ has a cusp of rank $r \implies$

$$\dim \Lambda > \frac{r}{2}.$$

Proof 1. (Case $r = 1$)

- Assume $(z \mapsto z + n, n \in \mathbb{Z}) \subset \Gamma$.
- (Spherical diameter near $x \in \mathbb{R}^n$)

$$\asymp \frac{\text{(Euclidean diameter near } x)}{|x|^2}.$$

- Limit set is periodic: $\Lambda = \bigcup_{\mathbb{Z}} \Lambda_n$.
- For D-dimensional measure:

$$\begin{aligned} m(\Lambda) &= \sum m(\Lambda_n) \asymp \sum \text{(Spherical diameter)}(\Lambda_n)^D \\ &\asymp \sum \frac{1}{n^{2D}} < \infty. \end{aligned}$$

- Therefore we have $D > 1/2$.

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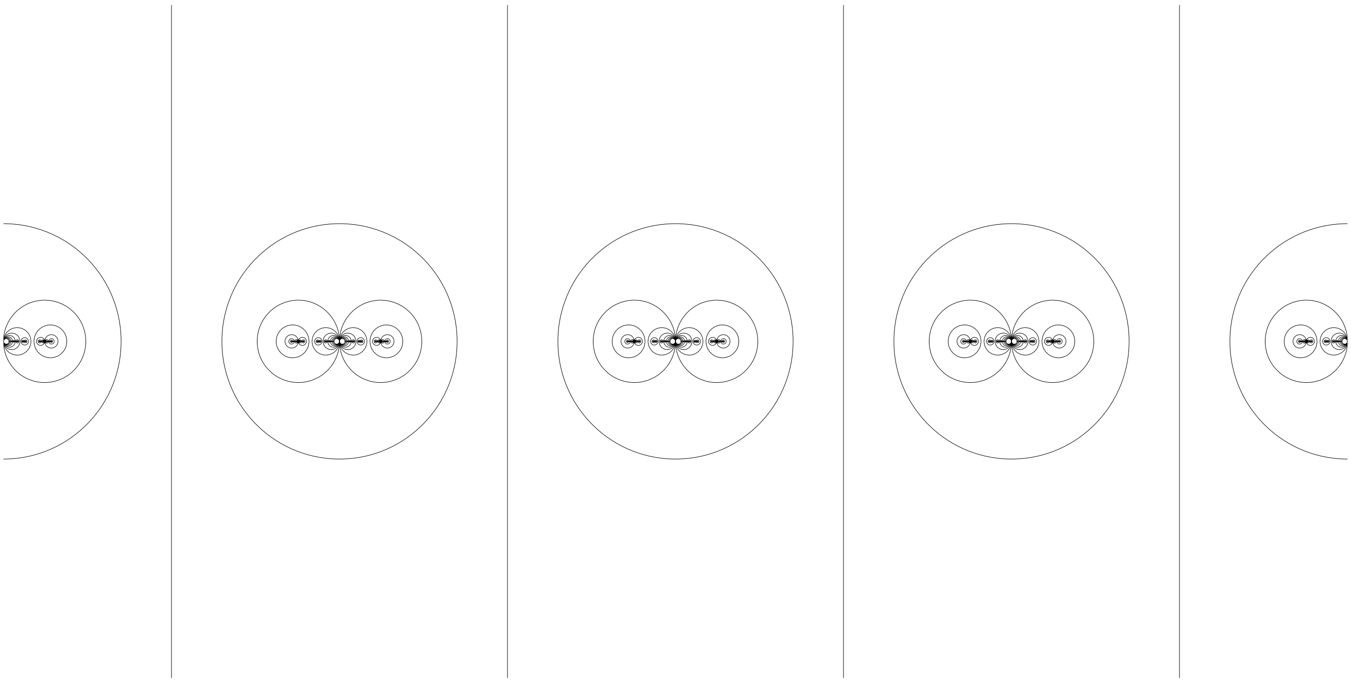
Proof 2. (Case of a Riemann surface)
The cusp supports eigenfunctions with $\lambda \to 1/4$.

Reflection group with a cusp



Theorem. $\dim \Lambda > 1/2$ even as $R \to 0$!

Dynamics of complex quadratic polynomials

Quadratic map: $f_c : \mathbb{C} \to \mathbb{C}$,

$$f_c(z) = z^2 + c.$$

Julia set:

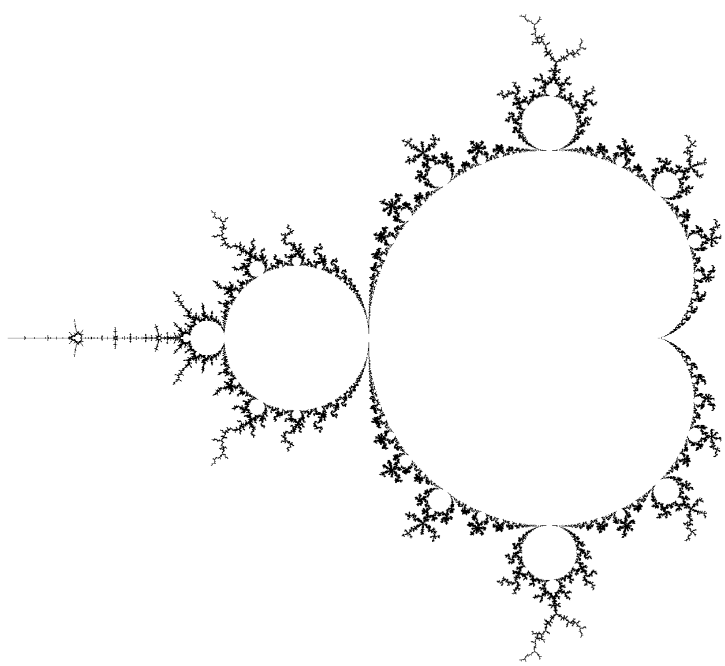
$$\begin{aligned} J(f_c) &= \partial\{z \ : \ \text{the orbit } z, f_c(z), f_c^2(z), f_c^3(z), \ldots \text{ is bounded.}\} \\ &= (\text{limit points of } f_c^{-n}(z_0)). \end{aligned}$$

($\rightsquigarrow$ the limit set Λ.)

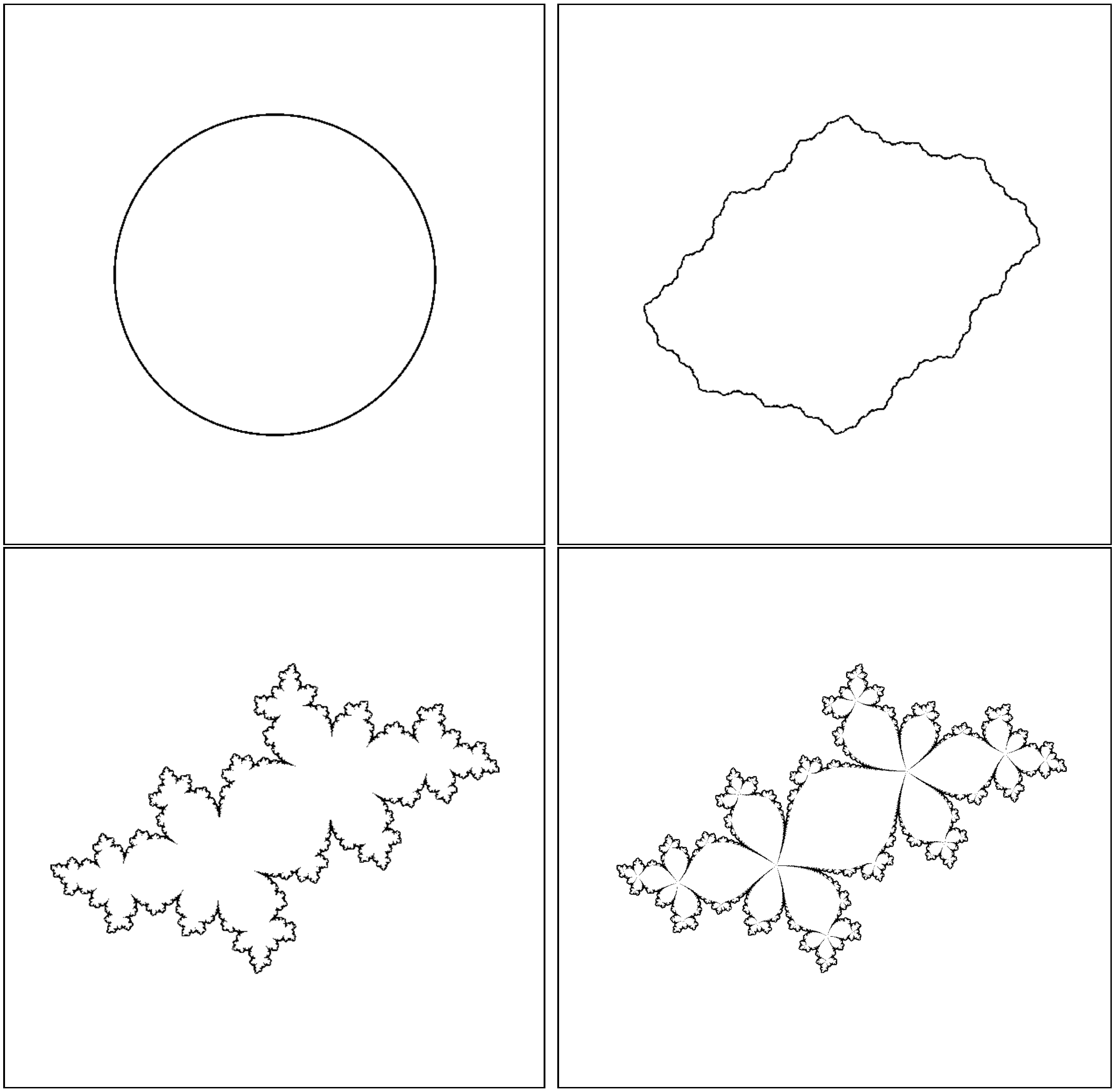
Mandelbrot set:

$$\partial M \quad = \quad \{c \in \mathbb{C} \ : \ \text{topology of } J(f_c) \text{ changes at } c\}.$$

∂M is like a catalog of Julia sets.



Birth of a parabolic Julia set



Rotation number $p/q = 2/5$

Rotation number and dimension

Theorem. $f(z) = \lambda z + z^2$ with λ a primitive qth root of 1 $\implies$

$$\dim J(f) \geq \frac{q}{q+1}.$$

Proof.

- q petals $\implies$ qth root of a cusp.
- Blocks J_n of Euclidean diameter $n^{1/q-1}$ near $n^{1/q} \in \mathbb{Z}^{1/q}$.
- (Spherical diameter J_n) $\asymp$

$$\frac{(\text{Euclidean diameter})}{n^{2/q}} \asymp n^{-(q+1)/q}$$

- $$\sum (\operatorname{diam} J_n)^D \asymp \sum n^{-D(q+1)/q} < \infty$$

$$\implies D > \frac{q}{q+1}.$$

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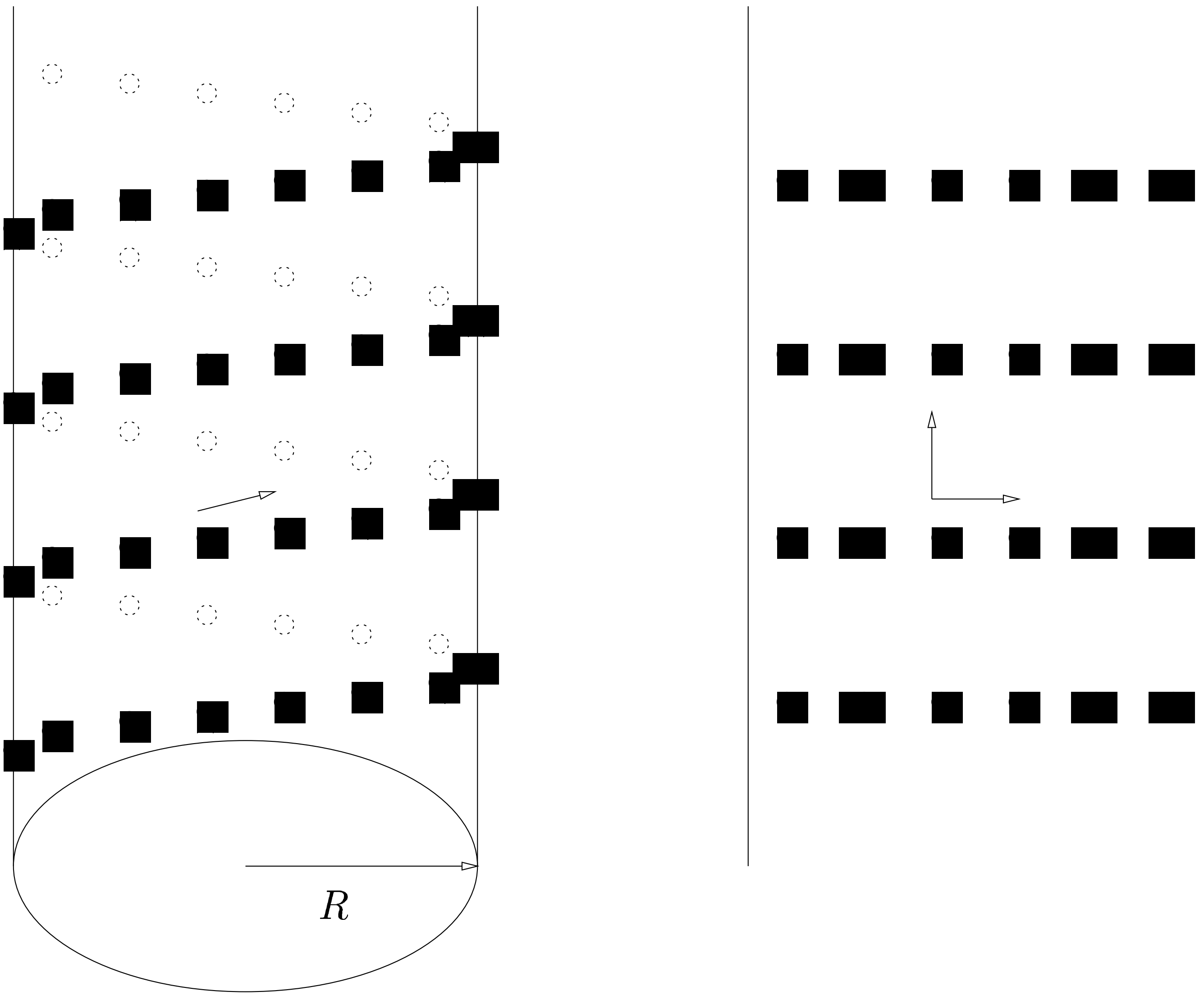
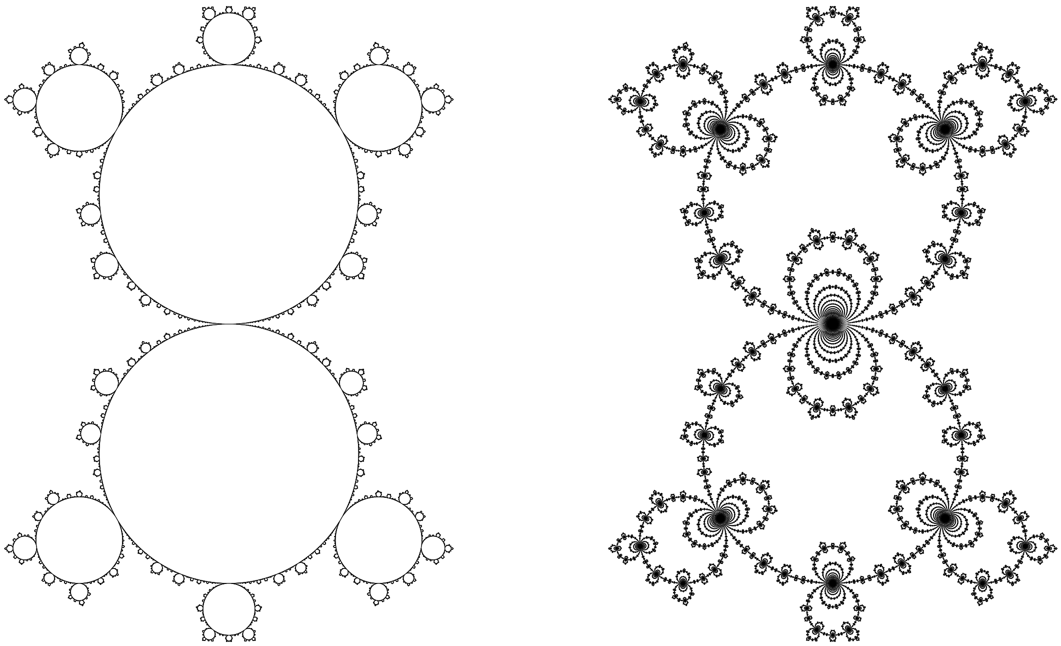
Geometric limits

$$\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

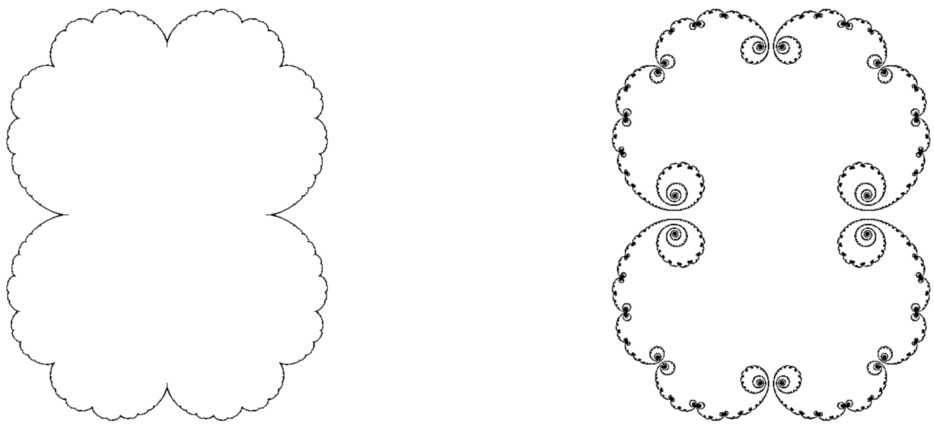


As $R \to \infty$, the screw motion converges to a lattice of translations.

Cusps of rank 1 and 2



Parabolic explosion



$$f(z) = z^2 + 1/4 \quad \text{and} \quad f(z) = z^2 + 1/4 + \epsilon$$

Rank 1 $\Longrightarrow$ Rank 2

Explosion and dimension

Theorem. Explosion of p/q-petals $\implies$

$$\lim \dim J(f_n) \geq \frac{2q}{q+1} \to 2$$

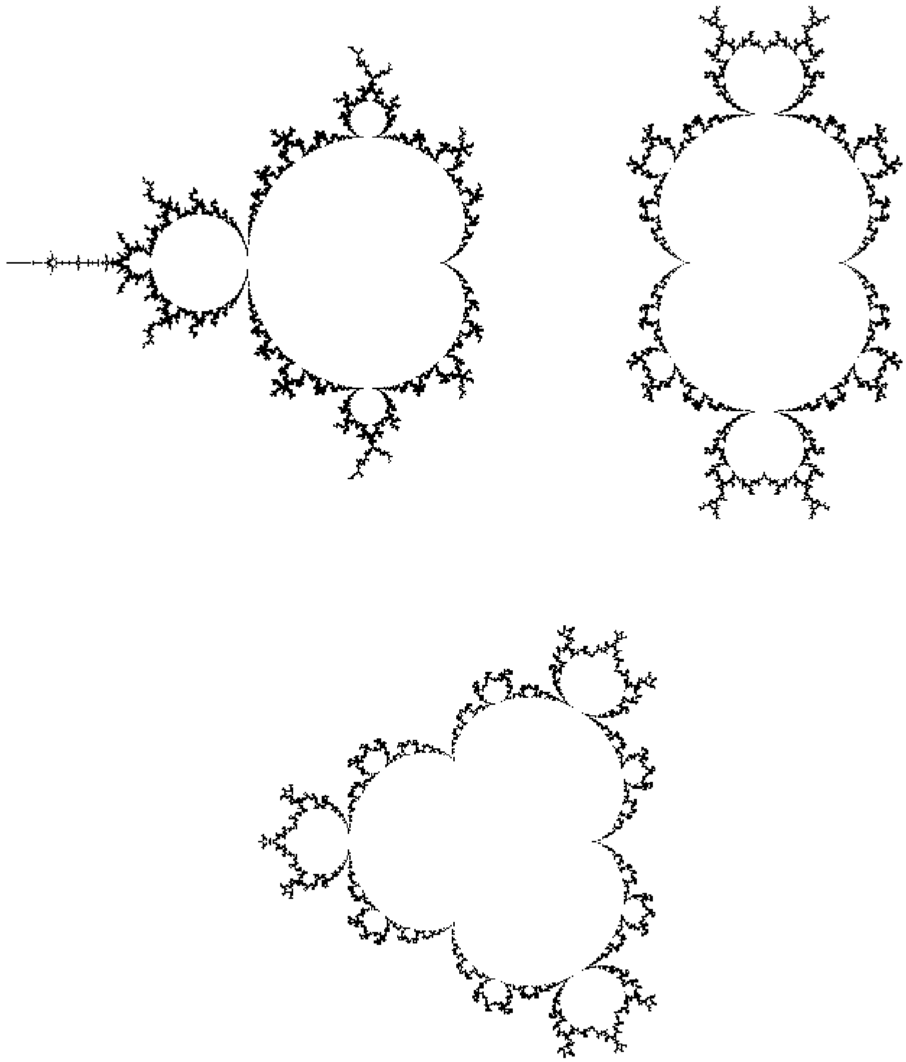
as $q \to \infty$.

Corollary. There exist $f(z) = z^2 + c$ with $\dim J(f) = 2$.

Corollary. ∂M looks like a catalog of Julia sets, so $\dim \partial M = 2$.

Open problem. Does the boundary of the Mandelbrot set have measure zero?

Mandelbrot's cousins



$M_d =$ Mandelbrot set for $z \mapsto z^d + c$.

Universality

Theorem. Every family of rational maps

$$f_t(z), \quad t \in \mathbb{C},$$

contains a copy of the Mandelbrot set, or its cousins, in its bifurcation locus.

Example. Newton's method for cubic polynomials:

$$f_t(z) = z - \frac{p(z)}{p'(z)},$$

where

$$p(z) = (z+1)(z-1)(z-t).$$

Corollary. The fractal dimension of every bifurcation set is as large as possible.

Newton's method