Riemann Surfaces in

Dynamics, Topology and Arithmetic

II. The shape of moduli space

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Dynamical systems

60's optimism. Conjecture: for any compact manifold M, the set of well-understood dynamical systems

$$f : M \to M$$

is open and dense in $\mathrm{Diff}(M)$.

Inspiration: triumphs of transversality and differential topology.
But false!

70's realism. Is there at least one map f with comprehensible dynamics in each component of $\mathrm{Diff}(M)$?

Surfaces

S = smooth, oriented closed surface of genus $g \geq 0$.

Mapping class group:

$$\begin{aligned} \mathrm{Mod}(S) &= (\text{diffeomorphisms of } S)/\text{isotopy} \\ &= (\text{components of } \mathrm{Diff}(S)) \end{aligned}$$

Goal:

Classify elements of $\mathrm{Mod}(S)$

Find *understandable* representatives in each isotopy class

Method: Find spaces on which $\mathrm{Mod}(S)$ acts.

Simple closed curves

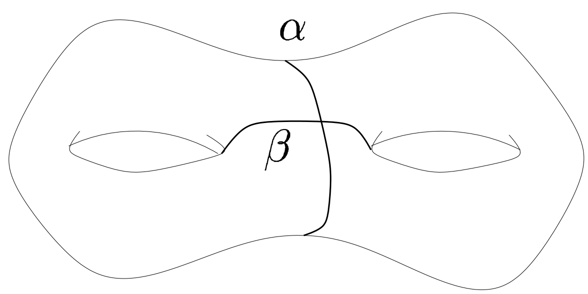
$\mathcal{S}$ = (simple closed curves on S)/isotopy

Intersection pairing:

$$i : \mathcal{S} \times \mathcal{S} \to \{0, 1, 2, 3, 4, \ldots\};$$

$i(\alpha, \beta)$ = minimal number of intersections possible.

Example:



$$i(\alpha, \beta) = 2$$

***Mod*(*S*) acts on $\boldsymbol{\mathcal{S}}$ preserving $\boldsymbol{i(\cdot, \cdot)}$**

The case of a torus (genus 1)

- $\mathrm{Mod}(S)$ acts faithfully on $H_1(S, \mathbb{Z}) \cong \mathbb{Z}^2$
- $\mathrm{Mod}(S) \cong SL_2(\mathbb{Z})$
- $\mathcal{S} = \mathbb{P}H_1(S, \mathbb{Q}) \cong \mathbb{P}(\mathbb{Q}^2) \subset \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$
- $\mathrm{Mod}(S)$ acts on $\mathcal{S}$ by Möbius transformations, preserving

$$i\left(\frac{p}{q}, \frac{r}{s}\right) = \left|\det \begin{pmatrix} p & r \\ q & s \end{pmatrix}\right|.$$

Classification of maps on a torus

The three types of $f \in \text{Mod}(S)$, for S a torus:

1. **Finite order.**

   $f_* : H_1(S, \mathbb{R}) \to H_1(S, \mathbb{R})$ has complex eigenvalues;
   $f^n = \text{id}$, some $n > 1$.

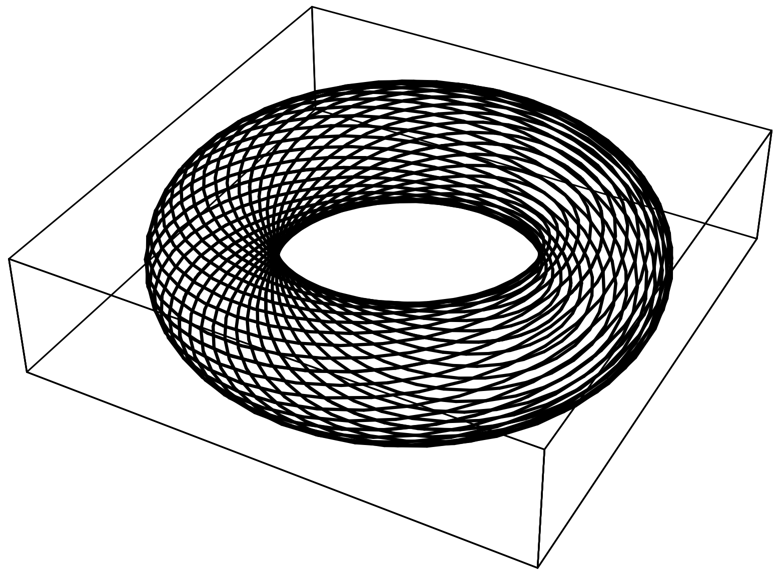
2. **Reducible.**

   f_* has a multiple eigenvalue (± 1);
   $f(\alpha) = \alpha$, some $\alpha \in S$;
   f is a Dehn twist around α.

3. **Anosov.**

   f_* has real eigenvalues $K^{\pm 1}$;
   $i(f^n(\alpha), \beta) \asymp K^n \to \infty$;
   $f \sim F$, an **area preserving, linear** map;
   F preserves a pair of **irrational foliations** of S.

Anosov example



The curve $F^4(\alpha)$ for $\alpha = (1, 0)$ and $F = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Higher genus

Theorem (Classification of Surface Diffeomorphisms)
Any $f \in \mathrm{Mod}(S)$ is finite order, reducible, or pseudo-Anosov.

—Thurston, 1979; Teichmüller, Nielsen, Bers

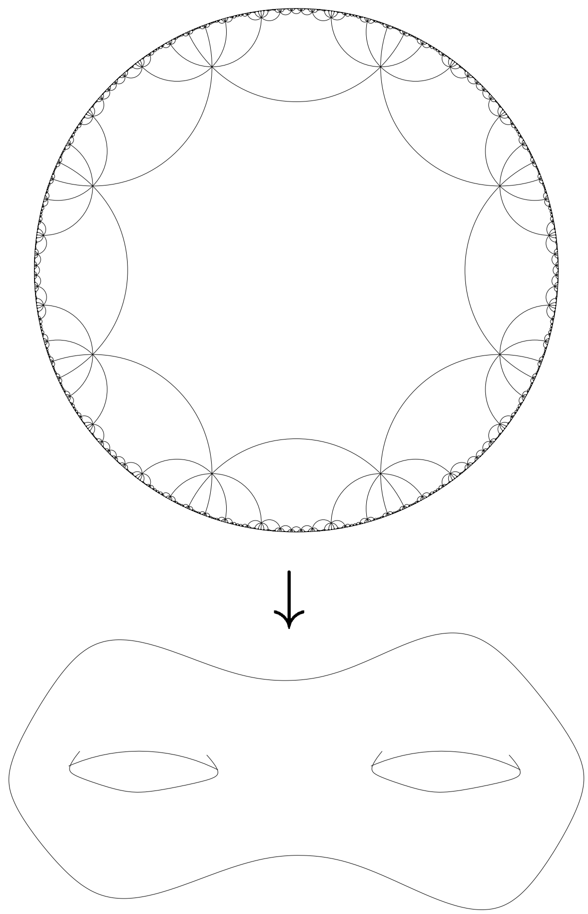
Reducible: there are disjoint simple curves $\alpha_1, \ldots, \alpha_n \in \mathcal{S}$ permuted by f.

Pseudo-Anosov: for some $K > 1$, $i(f^n(\alpha), \beta) \asymp K^n$.

(In fact $f \sim F$ locally like $F(x, y) = (Kx, K^{-1}y)$, preserving a pair of orthogonal foliations on S.)

Hyperbolic Riemann surfaces

Universal cover.
Any compact Riemann surface X of genus $g \geq 2$ is covered by the unit disk Δ.



Hyperbolic metric. X has a natural metric of curvature -1, coming from the invariant metric $2|dz|/(1-|z|^2)$ on the disk.

Finiteness of automorphisms

Theorem *For any compact Riemann surface X of genus $g \geq 2$, the group of holomorphic maps $F : X \to X$ is finite.*

Proof.

- F is a hyperbolic isometry $\implies$ Aut(X) is compact.
- If F_1 is close to F_2, then F_1 and F_2 are homotopic.
- Then $\widetilde{F_1} = \widetilde{F_2}$ on S^1, since

$$\frac{\text{(Euclidean metric)}}{\text{(Hyperbolic metric)}} \to 0 \qquad \text{at } \partial\Delta = S^1.$$

$\implies F_1 = F_2$.

- Thus Aut(X) is discrete.
- Discrete + compact $\implies$ finite.

∎

Corollary *If $F : X \to X$ is conformal, then F has finite order in* Mod(F)*.*

Teichmüller space

Teich($\boldsymbol{S}$) = (space of Riemann surfaces X marked by S).

A **marking** of X by S is a homotopy class of homeomorphism

$$h : S \to X.$$

Theorem Teich(S) *is homeomorphic to* $\mathbb{R}^{6g-6}$.

Mod(S) acts on Teich(S) by changing the marking:

$$f \circlearrowright \; S \xrightarrow{h} X.$$

Moduli space:

$$\begin{aligned}\mathcal{M}(S) &= \mathcal{M}_g \\ &= \mathrm{Teich}(S)/\,\mathrm{Mod}(S) \\ &= (\text{space of all Riemann surfaces } X \cong S).\end{aligned}$$

$$\pi_1(\mathcal{M}(S)) = \mathrm{Mod}(S)$$

Strategy

To analyze $f \in \mathrm{Mod}(S)$,

Search for a fixed-point in $\mathrm{Teich}(S)$.

f has a fixed-point $X \in \mathrm{Teich}(S) \implies$
$f \sim F \in \mathrm{Aut}(X) \implies$
f has finite order.

Short geodesics

Length function $\ell : \mathcal{S} \times \mathrm{Teich}(S) \to \mathbb{R}$:

$$\ell_\alpha(X) = (\text{length of closed geodesic} \sim \alpha \text{ on } X).$$

Theorem *There are at most $3g-3$ short geodesics on X, and they are all simple and disjoint.*

Theorem (Mumford) *The set of Riemann surfaces X in moduli space $\mathcal{M}(S)$ with*

$$\inf_\alpha \ell_\alpha(X) \geq r > 0$$

is **compact**.

Maps with minimal squeeze

Theorem (Teichmüller) *For any* $X, Y \in \mathrm{Teich}(S)$*, there exists a unique map*

$$G : X \to Y,$$

respecting markings, with minimal conformal distortion.

Local form:

$$G(x + iy) = Kx + iK^{-1}y$$

in suitable complex coordinates ($K > 1$).

Teichmüller metric on $\mathrm{Teich}(S)$:

$$d(X, Y) = \frac{1}{2}\log K.$$

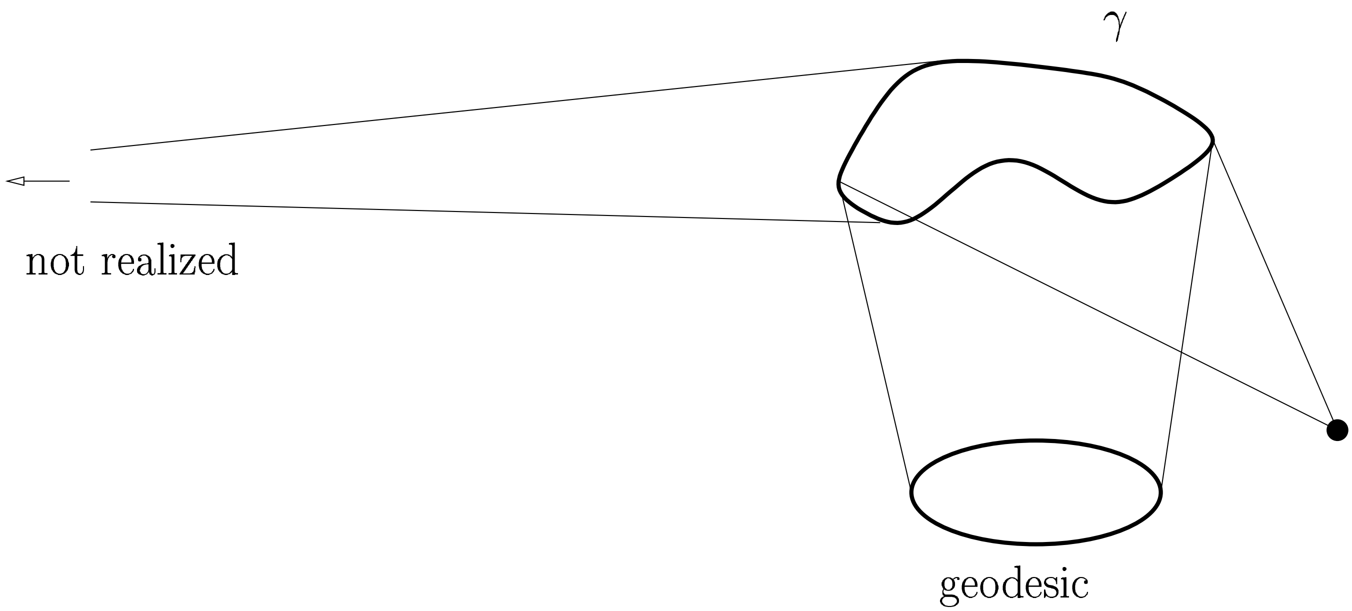
$\mathrm{Mod}(S)$ acts by isometries on $(\mathrm{Teich}(S), d)$.

Proof of the classification

Length of a mapping class:

$$\tau(f) \quad = \quad \inf\{\text{length}(\gamma \subset \mathcal{M}(S)) \ : \ \gamma \sim [f] \in \pi_1 \mathcal{M}(S)\}.$$

3 cases:



Geometric cases

Case 1. $\tau(f) = 0$, achieved. Then:

$f \cdot X = X$ for some X; $\implies$ f is of finite order.

Case 2. $\tau(f) > 0$, achieved. Then:

$F : X \to X$, Teichmüller map with $K(F)$ minimized, **preserves** its stretch foliations; $\implies$

$$i(f^n(\alpha), \beta) \asymp K^n$$

and $f \sim F$ is pseudo-Anosov.

Case of short geodesics

Case 3. $\tau(f)$ not achieved. Then:

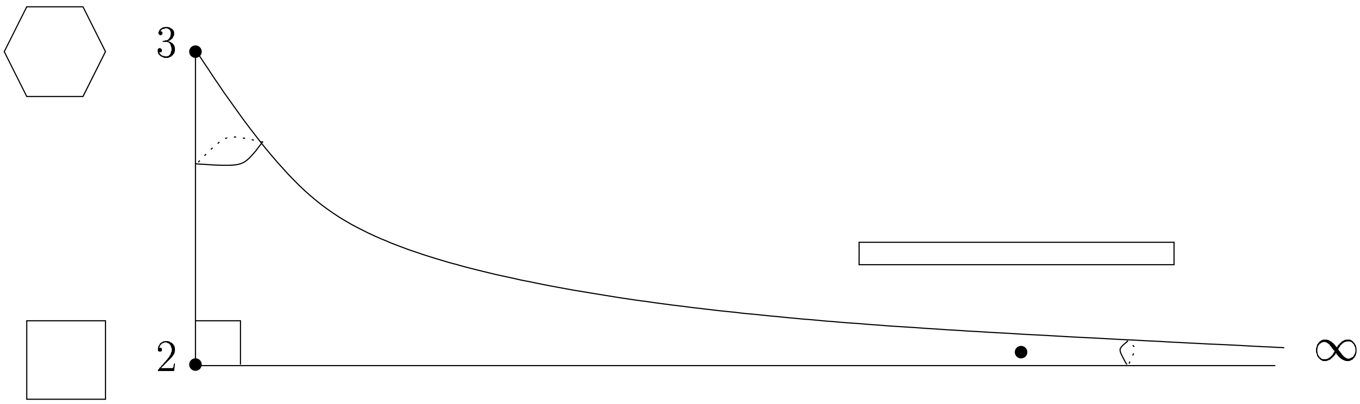
- We have loops $\gamma_n \to \infty$ in moduli space $\mathcal{M}(S)$;
- length$(\gamma_n) \approx \tau(f)$, $\gamma_n = [f] \in \pi_1$.
- Mumford's compactness theorem $\implies$ for $n \gg 0$, $X_n \in \gamma_n$ has disjoint short geodesics

$$\{\alpha_1, \ldots, \alpha_m\} \subset \mathcal{S};$$

- length(γ_n) bounded $\implies$ same geodesics short on $f \cdot X_n$
- $\implies$ f is reducible.

∎

Reprise: Moduli space for genus $g = 1$



Moduli space versus hyperbolic space

Let G be a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$.

Classification of isometries. Any $g \in G$ is either

1. **Elliptic:**
   g has a fixed-point in $\mathbb{H}^n \implies g$ is of finite order; or
2. **Hyperbolic:**
   g translates along a geodesic $\mathbb{H}^n$; or
3. **Parabolic:**
   g has a unique fixed-point $p \in S^{n-1}_\infty = \partial\mathbb{H}^n$.

- Compare finite order, pseudo-Anosov and reducible.
- Just as $\partial\mathbb{H}^n = S^{n-1}_\infty$, we have

$$\partial\,\mathrm{Teich}(S) = \mathbb{P}\mathcal{ML}(S) \cong S^{6g-7}.$$

Failure of hyperbolicity

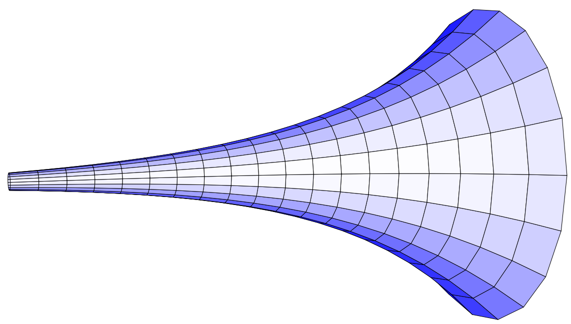
Theorem *For $g > 1$, the moduli space $\mathcal{M}_g$ admits no metric of pinched negative curvature* $(-a < K < -b < 0)$.

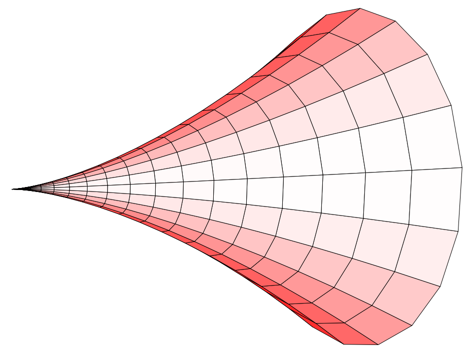
Proof. The fundamental group $\pi_1(\mathcal{M}_g)$ is generated by Dehn twists $(\tau_1, \ldots, \tau_n)$ with

$$[\tau_i, \tau_{i+1}] = 1.$$

For a negatively curved manifold this implies π_1 is virtually abelian, a contradiction. ∎

Metrics on Teichmüller space





Teichmüller metric

- Complete
- $\text{vol}(\mathcal{M}_g) < \infty$
- Curvature -1 when $g = 1$

- Not Riemannian $(g > 1)$

Weil-Petersson metric

- Kähler
- Convex
- Curvature ≤ 0

- Curvature $\to -\infty$
- Incomplete

Kähler hyperbolicity

Theorem *The Teichmüller metric on moduli space $\mathcal{M}_g$ is comparable to a Kähler hyperbolic metric h.*

$\Longrightarrow$ We have a symplectic form ω on $\mathcal{M}_g$ such that:

- The corresponding Kähler length satisfies:

$$\|v\|_h^2 = \omega(v, Jv) \asymp \|v\|_{\text{Teich}}^2;$$

- $\widetilde{\omega} = d\theta$ for a bounded 1-form θ on $\text{Teich}(S)$; and
- $(\mathcal{M}_g, h)$ is complete, with finite volume and bounded curvature.

Consequences of Kähler hyperbolicity

In the Teichmüller metric:

- The least eigenvalue of the Laplacian satisfies
$$\lambda_0(\mathrm{Teich}(S)) > 0.$$
- For any compact *complex* submanifold $N^{2k} \subset \mathrm{Teich}(S)$,
$$\mathrm{vol}(N) \leq C \cdot \mathrm{vol}(\partial N).$$
- L^2-cohomology of $\mathrm{Teich}(S)$ lives in the middle dimension.
- The Euler characteristic satisfies:
$$\mathrm{sign}\,\chi(\mathcal{M}_g) = (-1)^{\dim_{\mathbb{C}} \mathcal{M}_g}.$$

The $1/\ell$ metric

Bers' embedding:

$$\begin{aligned} \beta_X : \operatorname{Teich}(S) &\to Q(X) \\ &\cong T_X^* \operatorname{Teich}(S) \\ &\cong \mathbb{C}^{3g-3}. \end{aligned}$$

Bounded 1-form on $\operatorname{Teich}(S)$**:**

$$\theta_Y(X) = \beta_X(Y) \in T_X^* \operatorname{Teich}(S).$$

Theorem *The Weil-Petersson Kähler form is d(bounded):*

$$\omega_{\mathrm{WP}} = d(i\theta_Y).$$

The $1/\ell$ metric:

$$\omega_{1/\ell} = \omega_{\mathrm{WP}} - i\delta \sum_{\ell_\alpha(X)<\epsilon} \partial\overline{\partial} \operatorname{Log} \frac{\epsilon}{\ell_\alpha}$$

Theorem *The $1/\ell$-metric is Kähler hyperbolic, and comparable to the Teichmüller metric.*

Compare: The $1/d$ metric.
On a smoothly bounded region $\Omega \subset \mathbb{C}$,

$$\rho_{1/d} = \frac{|dz|}{d(z, \partial\Omega)}$$

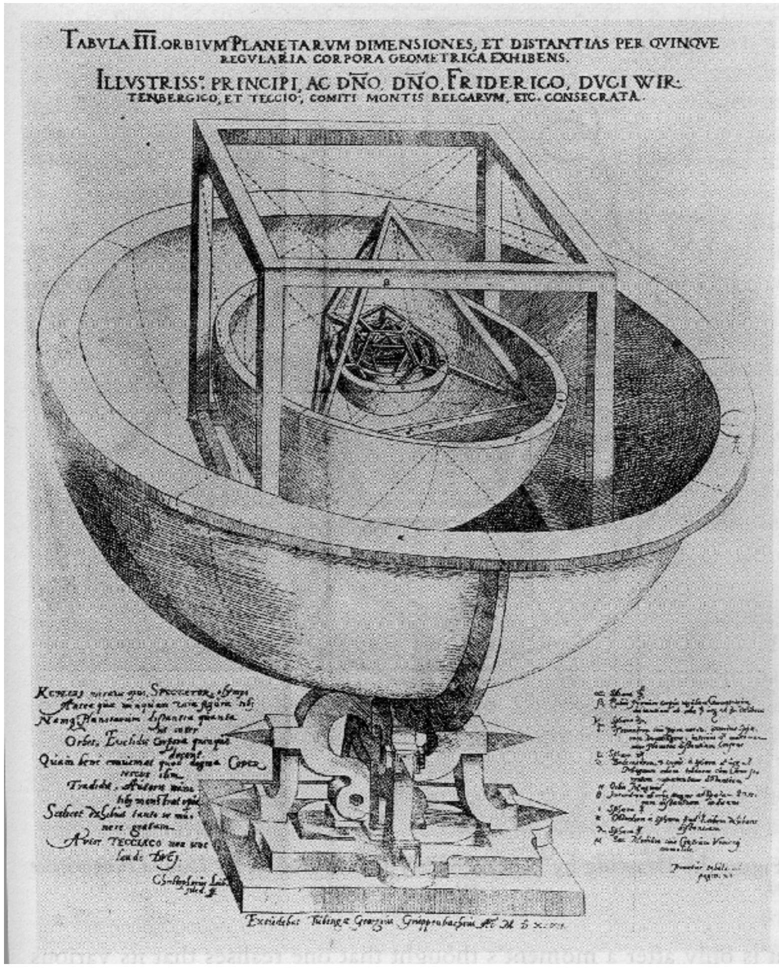
is comparable to the hyperbolic metric.

Kepler's orbs

	Predicted	Actual
Jupiter/Saturn	577	635
Mars/Jupiter	333	333
Earth/Mars	795	757
Venus/Earth	795	794
Mercury/Venus	577	723

—Kepler, *Mysterium Cosmographicum*, 1596.

Orbs


TABVLA III. ORBIVM PLANETARVM DIMENSIONES, ET DISTANTIAS PER QVINQVE REGVLARIA CORPORA GEOMETRICA EXHIBENS.
ILLVSTRISS. PRINCIPI, AC DNO. DNO. FRIDERICO, DVCI WIRTENBERGICO, ET TECCIO, COMITI MONTIS BELGARVM, ETC. CONSECRATA.

Next time

Instead of studying maps

$$S^1 \to \mathcal{M}_g,$$

we will study maps

$$X \to \mathcal{M}_g$$

where X is a Riemann surface.