From Platonic Solids To Quantum Topology

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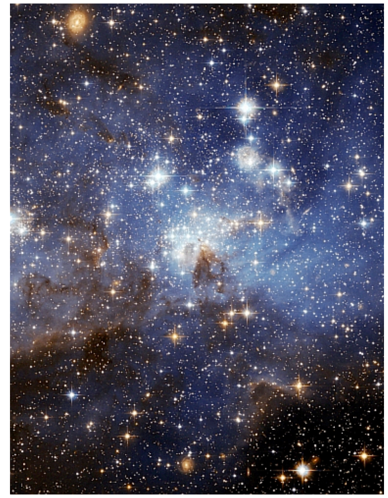
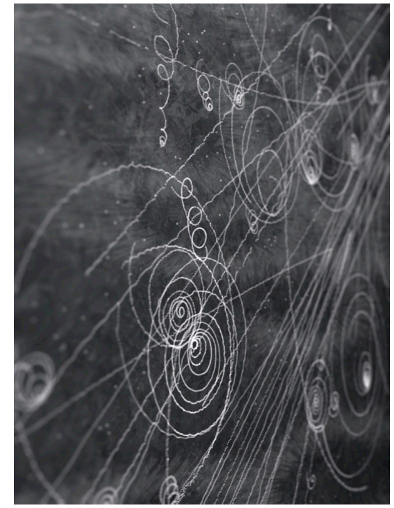
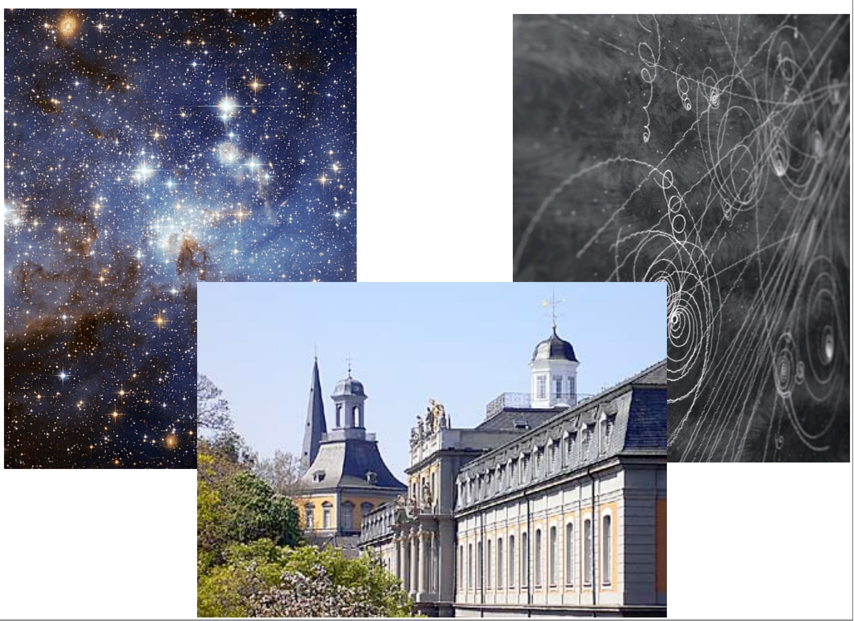
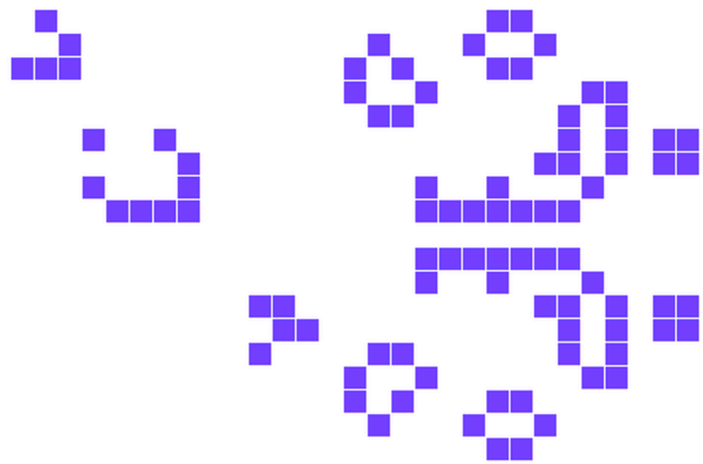
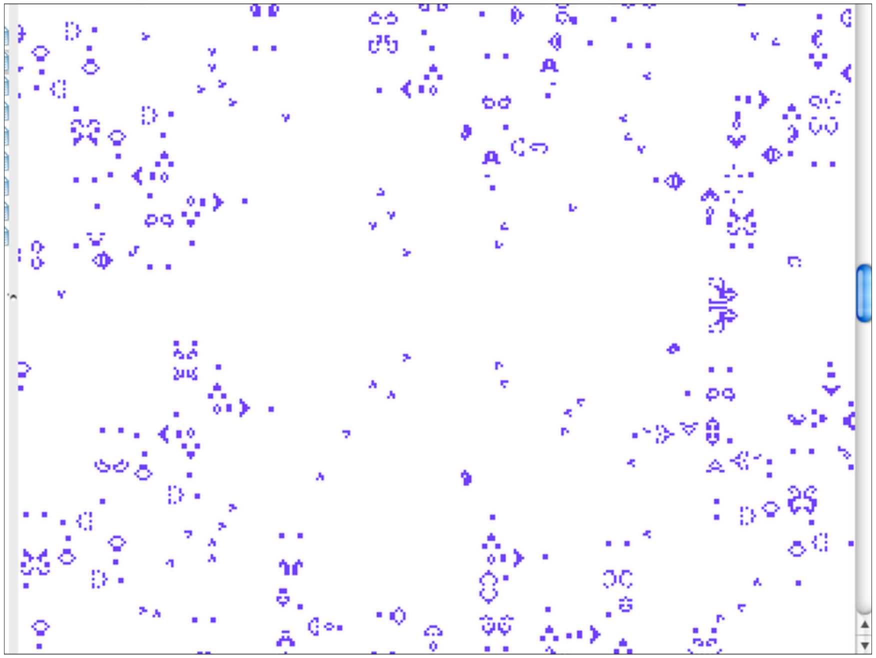
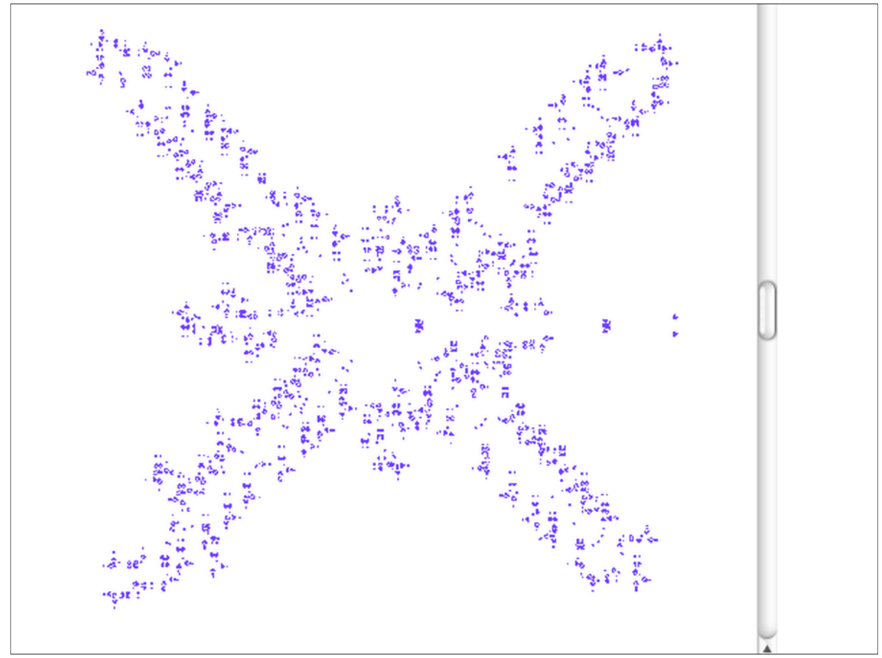


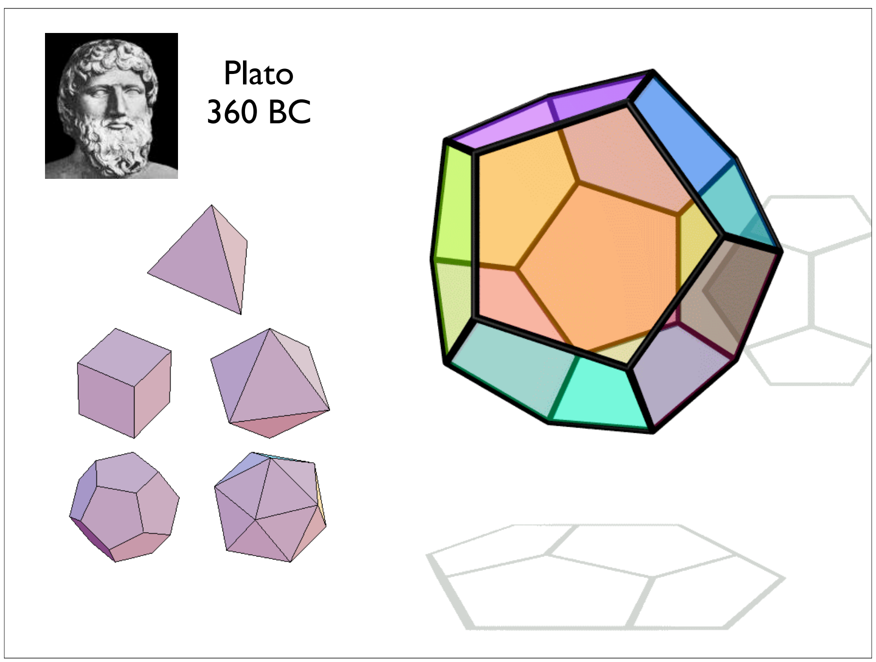


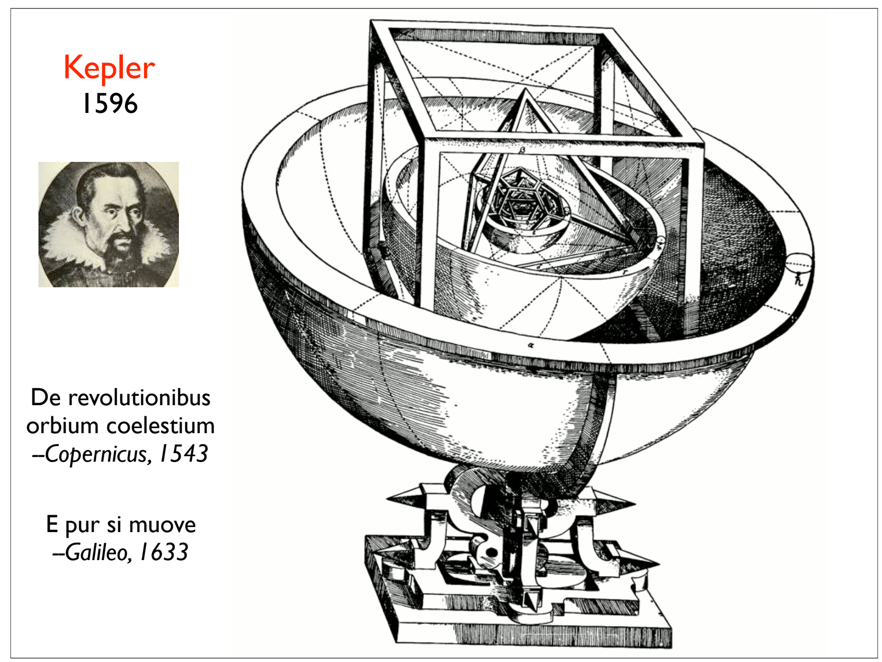


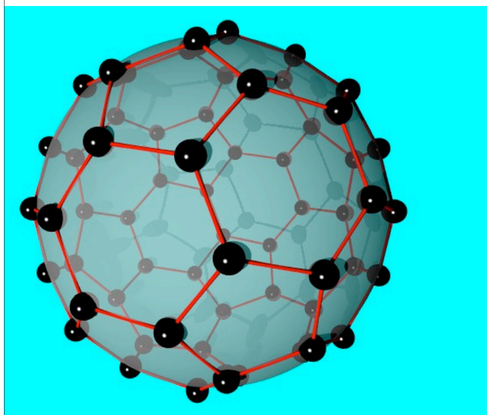


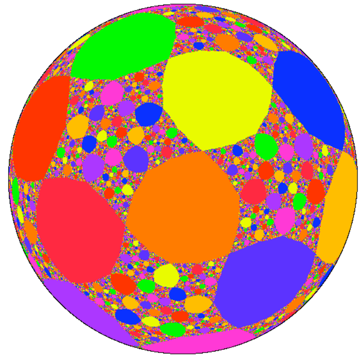





Plato
360 BC


Kepler
1596
De revolutionibus
orbium coelestium
–Copernicus, 1543
E pur si muove
–Galileo, 1633

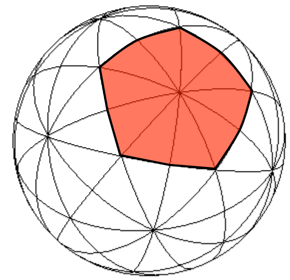


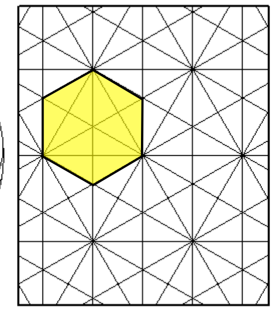


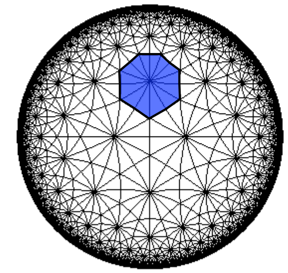
Buckminsterfullerene C_{60}
Kroto Curler Smalley 1985

Dynamical solution of quintics

Klein, 1879



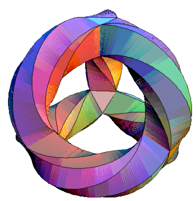


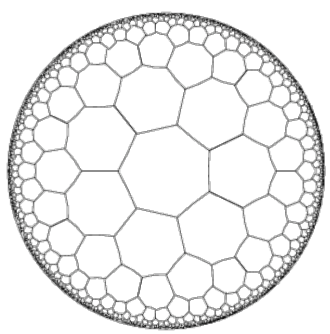


$PSL_2(Z/5)$
order 60

$PSL_2(Z/7)$
order 168

Genus 3





Helaman Ferguson, 1993

$168 = 7 \times 24 = |PSL_2(Z/7)|$

Braids

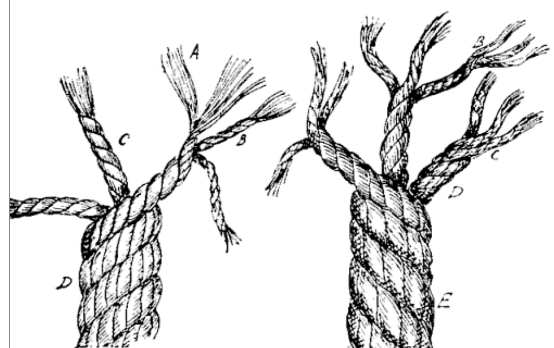
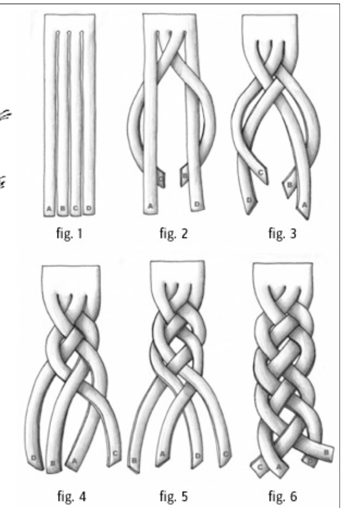


FIG. 1.—Construction of rope.



$P_n = \left\{ \begin{array}{l} \text{Polynomials } p(x) \text{ of degree } n \\ \text{with } \textit{distinct} \text{ roots} \end{array} \right\}$

Braid group
$B_n = \pi_1(P_n)$

Hodge theory

Polynomial ⇒ Riemann surface

$X : y^d = (x-b_1) \dots (x-b_n)$

$X \downarrow Z/d = \langle T \rangle$ P^1

⇒ complex cohomology group $H^1(X)$ intersection form

⇒ eigenspace $\mathrm{Ker}(T-qI) = H^1(X)_q$ (r,s) $q^d=1$

⇒ rep $\rho_q : B_n = \pi_1(P_n) \to U(H^1(X)_q)$ U(r,s)

⇒ period map $f_q : \mathcal{T}^*_{0,n} \to$ [positive lines in $H^1(X)_q$] $= \left\{ \begin{array}{l} \mathbb{CP}^{r-1} \\ \mathbb{CH}^s \\ \dots \end{array} \right.$

$[p(x)] \to [dx/y^k]$ $q = e(-k/d)$

n=3: Triangle groups

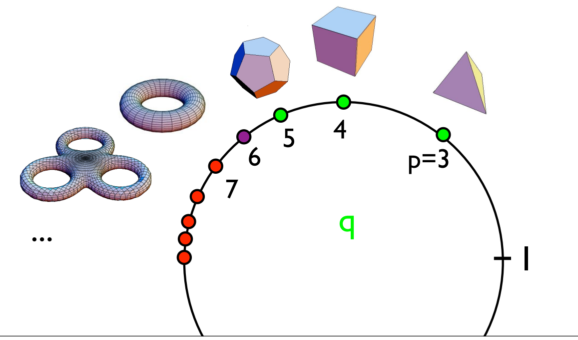
Polynomial ⇒ Riemann surface

$X : y^d = x(x-1)(x-t)$

$X \downarrow Z/d$ P^1

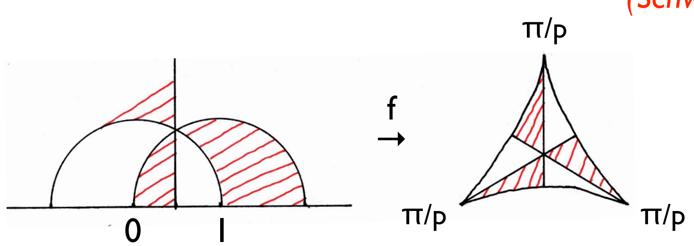
$\rho_q : B_3 \to U(2)$ or $U(1,1)$

$q = -e(-1/p)$



n=3: Period map = Riemann map

(Schwarz)



$$f(t) = \int_0^1 \frac{dx}{y^k} \Big/ \int_1^\infty \frac{dx}{y^k}$$

$$y^d = x(x-1)(x-t)$$

Classical Platonic [finite] case: Integral miracles

$\rho_q(B_4) \subset U(3)$ finite

$$\left(\int_0^1 \frac{dx}{(x(x-1)(x-a)(x-b))^{3/4}} \right)^4 =$$

$$\frac{\pi^2 \Gamma(1/4)^4}{\Gamma(3/4)^4} \cdot \frac{((2a-1)\sqrt{b(b-1)} + (2b-1)\sqrt{a(a-1)})^4}{a^2 b^2 (a-1)^2 (b-1)^2 (a+b-2ab-2\sqrt{ab(a-1)(b-1)})^3}$$

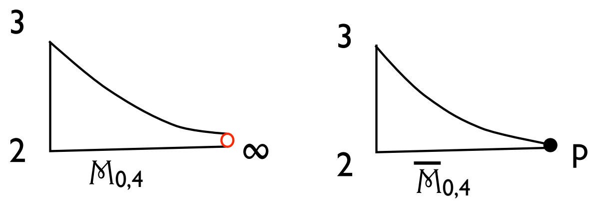
Note: upon changing 3/4 to 1/2, we obtain an elliptic modular function (transcendental).

Complex Hyperbolic Case

$\rho_q(B_n) \subset U(1,s) \to \mathrm{Isom}(\mathbb{CH}^s)$

Theorem (Deligne-Mostow, Thurston)

$\overline{\mathcal{M}}_{0,n}$ is a complex hyperbolic manifold for n=4,5,6,8,12.

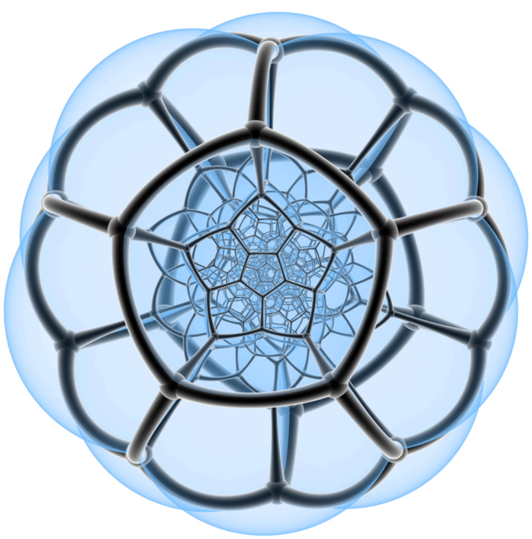


Theorem

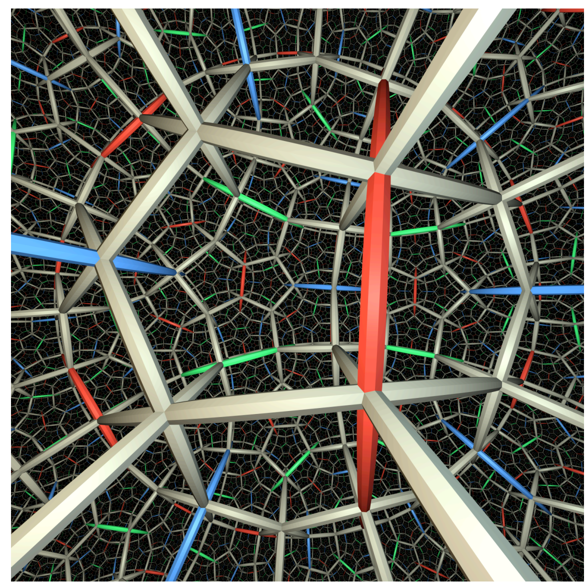
The Platonic solids and their generalizations are moduli spaces of points configurations on P^1, *made geometric* using Hodge theory.

- *Hypersurfaces of low degree in P^2, P^3, P^4, and $P^1 x P^1$ have been similarly investigated by branched covers/Hodge theory (Allcock, Carlson, Toledo, Kondo, Dolgachev, Looijenga, ...)*

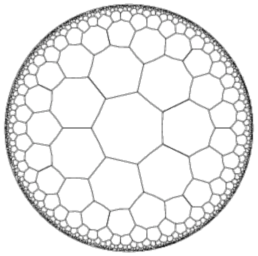
Positively Curved 3-Manifolds



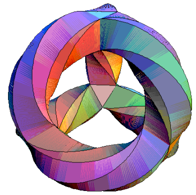
Negatively Curved 3-Manifolds

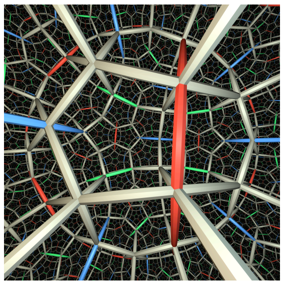


Tilings, Groups, Manifolds



Γ in $SL_2(R)$ = Isom(H^2)
H^2/Γ = surface



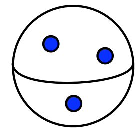


Γ in $SL_2(C)$ = Isom(H^3)
H^3/Γ = 3-manifold

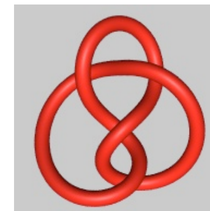
?

Arithmetic Examples

Γ = $SL_2(Z)$ $SL_2(Z[\omega])$ $SL_2(Z[i])$

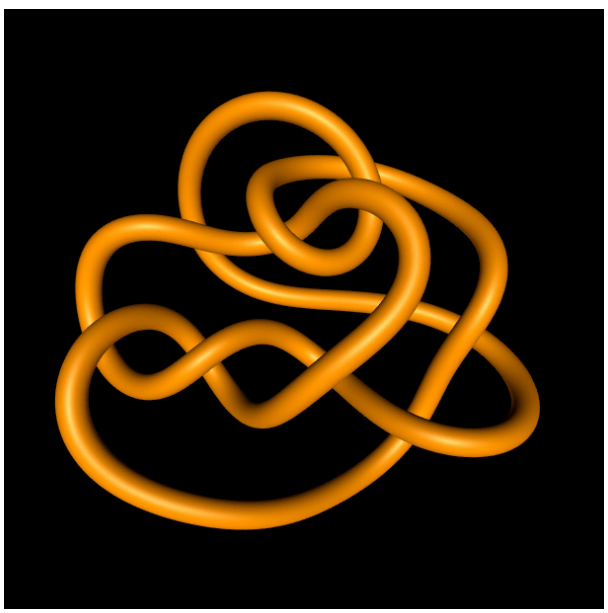


$$\mathbb{H}^2/\Gamma = S^2 \backslash K$$

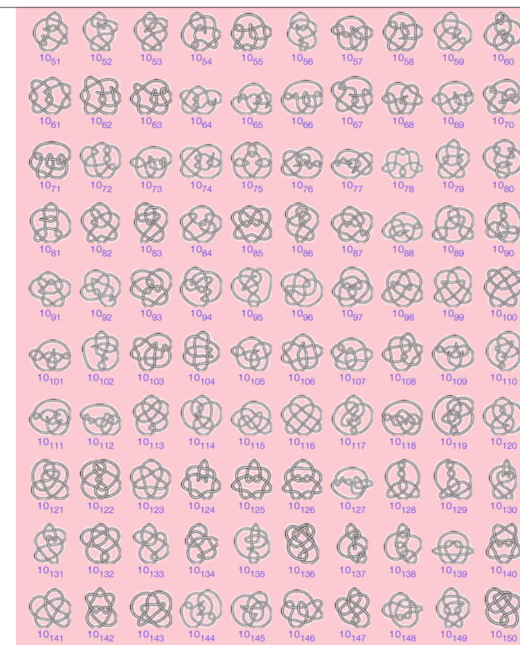


$$\mathbb{H}^3/\Gamma = S^3 \backslash K$$

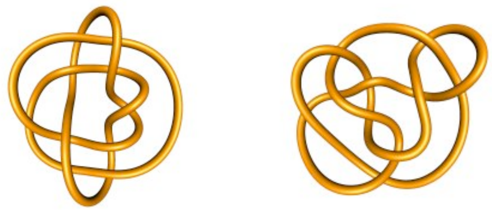
Mostow: Topology ⇒ Geometry



Knots with 10 crossings



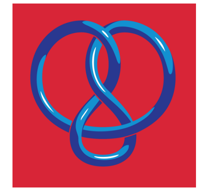
The Perko Pair

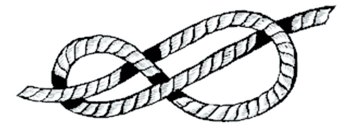


Hyperbolic volume

as a topological invariant

K =





$$\mathrm{vol}(S^3 - K) = 6Л(\pi/3) = 6\int_0^{\pi/3} \log \frac{1}{2\sin\theta}\, d\theta$$

$$= 2.0298832128193...$$

Thurston's theorem

1980s

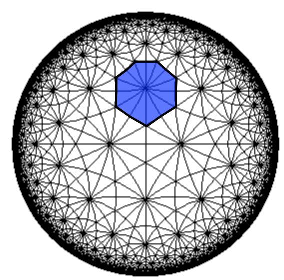
Almost all knot complements are hyperbolic.

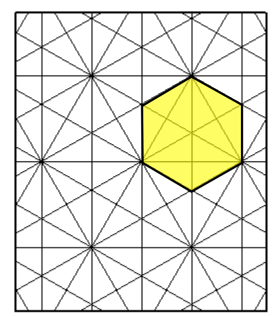
Hoste, Thistlethwaite and Weeks,1998:
The First 1,701,936 knots

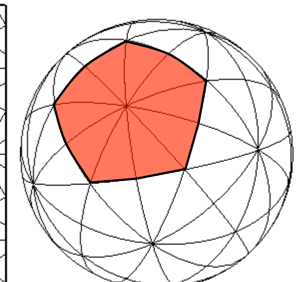
(Up to 16 crossings)

Towards the Poincaré Conjecture?

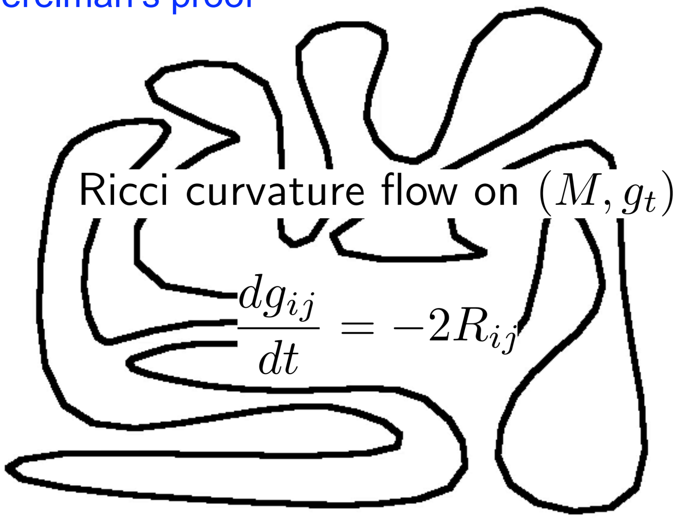
M^3-K	M^3-K	$M^3=S^3$
angle small	angle 180	angle 360







Perelman's proof



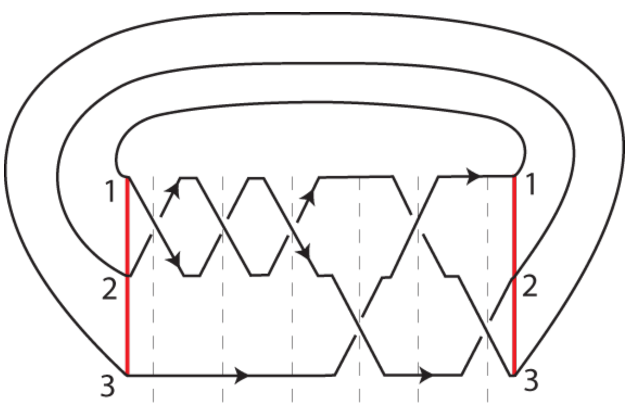
Ricci curvature flow on (M, g_t)

$$\frac{dg_{ij}}{dt} = -2R_{ij}$$

Theorem:

General relativity places no constraints on the topology of the Universe.

Quantum permutations + knots

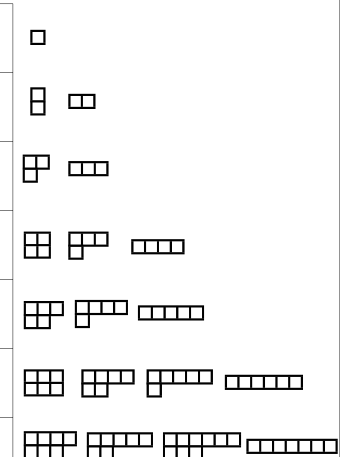


$\beta =$ a a a b a^{-1} b

permutation (123)

Representations of braid groups

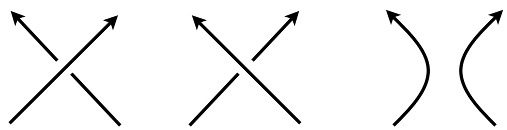
B_1		1						
B_2	1		1					
B_3		2		1				
B_4	2		3		1			
B_5		5		4		1		
B_6	5		9		5		1	
B_7		14		14		6		1



The Jones polynomial

For K *= the closure of* β *in* B_n:

$$V(K,t) = (-t^{1/2} - t^{-1/2})^{n-1}\, t^{\deg(\beta)}\, \mathrm{Tr}(\beta)$$

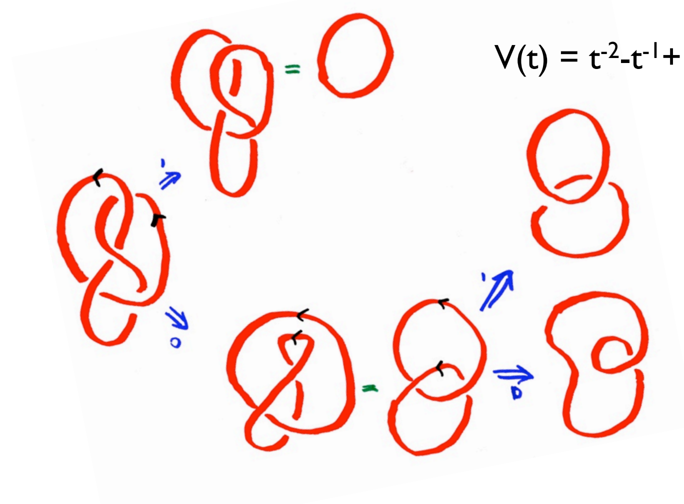


skein theory

$$t^{-1}V_+ \quad - \quad tV_- \quad = \quad (t^{1/2} - t^{-1/2})\, V_0$$

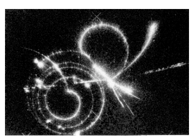
$V(O,t) = 1$

Jones polynomial for figure 8 knot



$V(t) = t^{-2}-t^{-1}+1-t+t^2$

Quantum fields



(Witten)

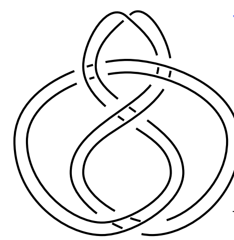
$$\langle K\rangle = \int \mathrm{Tr}(\oint_K A)\; e^{2\pi i k\, CS(A)}\; DA$$

$$= (q^{1/2}+q^{-1/2})\, V(K,1/q)$$

$q = \exp(2\pi i/(2+k)) \to 1$ *as* $k \to \infty$

$\langle \text{unknot}\rangle \to 2$

Volume Conjecture



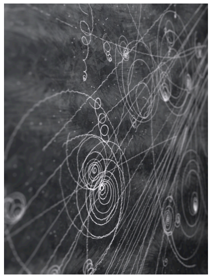
Cable K^2 *for figure eight knot* K

Murakami-Murakami
Kashaev

$$V_{n+1}(K,t) = \sum_{j=0}^{n/2} (-1)^j \binom{n-j}{j} V(K^{n-2j},t)$$

$$\frac{2\pi \ \log |V_n(K,e^{2\pi i/n})|}{n} \to \text{hyperbolic } \mathrm{vol}(S^3\text{-}K)$$

quantum fields — *general relativity*



Challenge:

Find the geometry of a 3-manifold using quantum topology