Slide 1

Surfaces in the space of surfaces

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coauthors Mukamel and Wright

Slide 2

Planes in hyperbolic manifolds

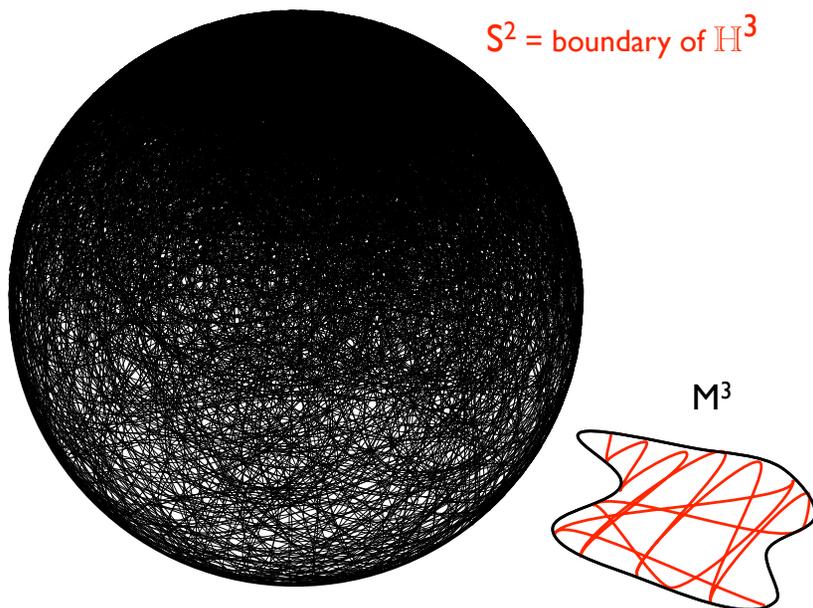$$f : \mathbb{H}^2 \to M^n = \mathbb{H}^n/\Gamma$$

Theorem (Shah, Ratner)

The closure of $f(\mathbb{H}^2)$ is a compact, immersed,

totally geodesic submanifold

N^k inside M^n.

Ex: $f : \mathbb{H}^2 \to M^3$
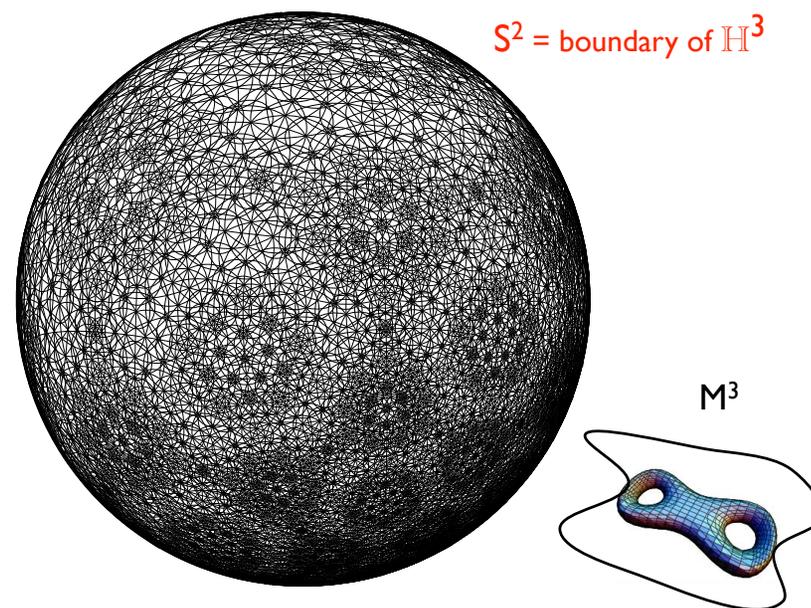
Im(f) = a closed surface, or
Im(f) is dense in M^3.

Slide 3

Dense plane in M^3

S^2 = boundary of $\mathbb{H}^3$
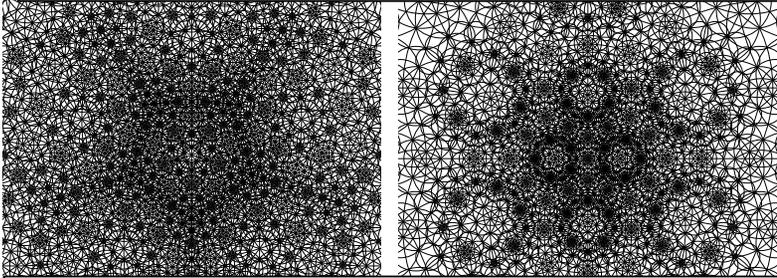
M^3

Slide 4
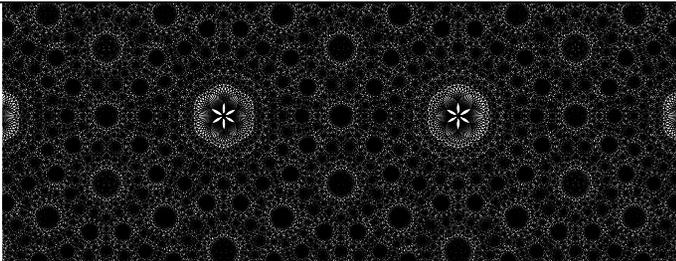
Closed, totally geodesic surface in M^3

S^2 = boundary of $\mathbb{H}^3$
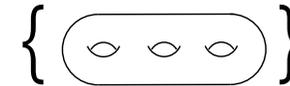
M^3

Arithmetic tetrahedra

Open problem: Do ∞ many closed geodesic surfaces ⇒ M is arithmetic?

Moduli space

$\mathcal{M}_g$ = moduli space of Riemann surfaces X of genus g
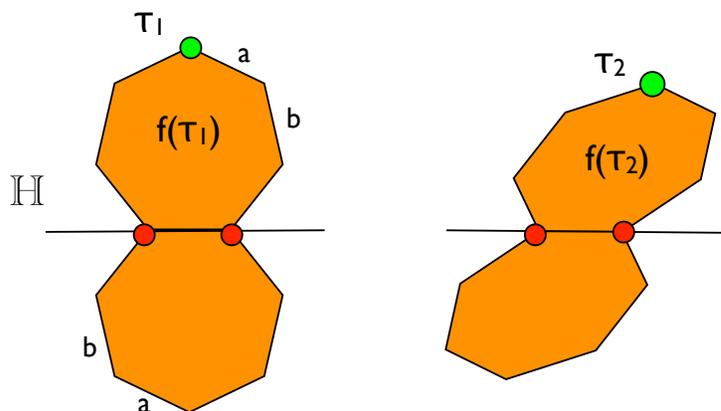

-- a complex variety, dimension 3g-3

Teichmüller metric

There exists a holomorphic, isometrically immersed
complex geodesic
$$f : \mathbb{H}^2 \to \mathcal{M}_g$$
through every point in every possible direction.

Example of a complex geodesic $f : \mathbb{H}^2 \to \mathcal{M}_3$

$f(\tau) = \text{Polygon}(\tau)/\text{gluing} = \text{genus 3 } X(\tau)$

Planes in $\mathcal{M}_g$

$$f : \mathbb{H}^2 \to \mathcal{M}_g = \mathcal{T}_g/\text{Mod}_g$$

Theorem (M, Eskin-Mirzakhani-Mohammadi, Filip)

The closure of $f(\mathbb{H}^2)$ is an algebraic subvariety of moduli space.

2002, 2014

Example: For g=2, the closure of $f(\mathbb{H}^2)$ can be
a Teichmüller curve,
a Hilbert modular surface,
or the whole space.

The Hilbert modular surface is not totally geodesic.

Totally geodesic subvarieties

$\mathcal{M}_g \subset \mathbb{P}^N$ is a projective variety

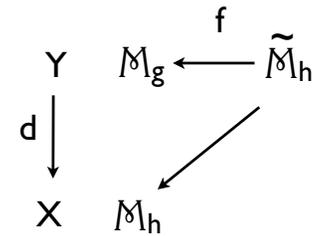Almost all subvarieties $V \subset \mathcal{M}_g$ are *contracted*.

PROBLEM

What are the *totally geodesic** subvarieties $V \subset \mathcal{M}_g$?

(*Every complex geodesic *tangent* to V is contained in V.)

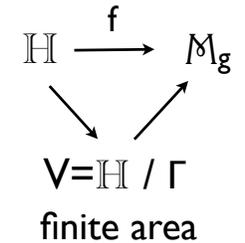## Known geodesic subvarieties in $\mathcal{M}_g$

I. Covering constructions

$$Y \qquad \mathcal{M}_g \xleftarrow{f} \widetilde{\mathcal{M}}_h$$
$$d \downarrow \qquad \qquad \searrow$$
$$X \qquad \mathcal{M}_h$$

Im(f) = a totally geodesic subvariety

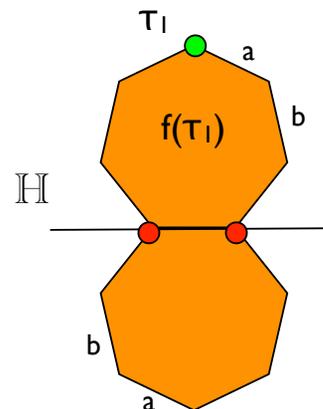*Example:* $\widetilde{\mathcal{M}}_{1,2} \to \mathcal{M}_{1,3}$

II. Teichmüller curves

$$\mathbb{H} \xrightarrow{f} \mathcal{M}_g$$
$$\searrow \qquad \nearrow$$
$$V = \mathbb{H} / \Gamma$$
$$\text{finite area}$$

Im(f) = a totally geodesic curve

Example of a Teichmüller curve

$$\mathbb{H} \xrightarrow{f} \mathcal{M}_3$$
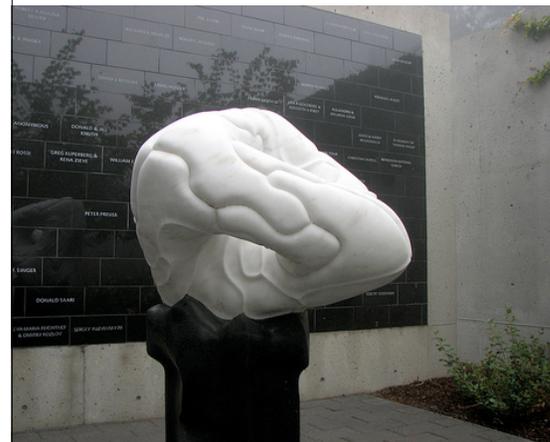$$\searrow \qquad \nearrow$$
$$V = \mathbb{H} / \Gamma(2,7,\infty)$$
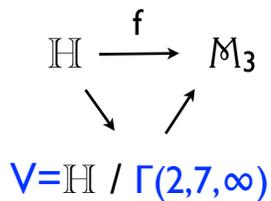
= the Klein quartic!

Klein quartic
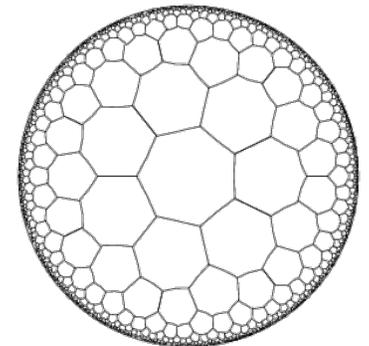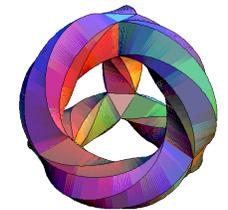
Helaman Ferguson, 1993

Thurston, MSRI director, 1992-1997

$168 = 7\times24 = |PSL_2(\mathbb{Z}/7)|$

Totally geodesic varieties in moduli space

Theorem

There is a primitive, totally geodesic complex surface F (the flex locus) properly immersed into $\mathbb{M}_{1,3}$.

Proof that F does not exist

Let V be a totally geodesic hypersurface in $\mathbb{M}_g$.

Given [X] in V, let $q_0,...q_n$ be a basis for $Q(X) = T^*_X \mathbb{M}_g$.

Assume q_0 generates the normal bundle to V.

Then the *highly nonlinear condition*:

$$\int_X q_0 \frac{\sum \overline{a}_i \overline{q}_i}{|\sum a_i q_i|} = 0$$

Is equivalent to a linear condition of the form

$$\sum a_i b_i = 0.$$

1st example of a Teichmüller surface

The flex locus $F \subset \mathbb{M}_{1,3}$
is the set of

(A,P) in $\mathbb{M}_{1,3}$:

$\exists$ degree 3 map $\pi: A \rightarrow \mathbb{P}^1$ such that

(i) P ~ Z = any fiber of π; and

(ii) P $\subset$ cocritical points of π.

What is the dimension of F? Is F irreducible? ....

1879

A TREATISE

ON THE
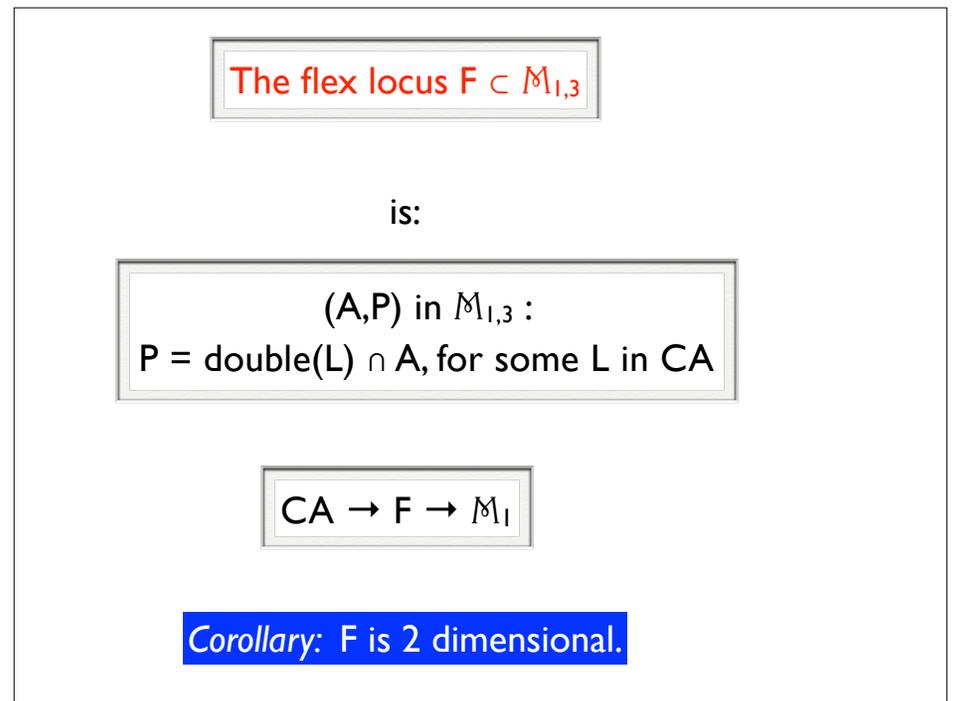
HIGHER PLANE CURVES:

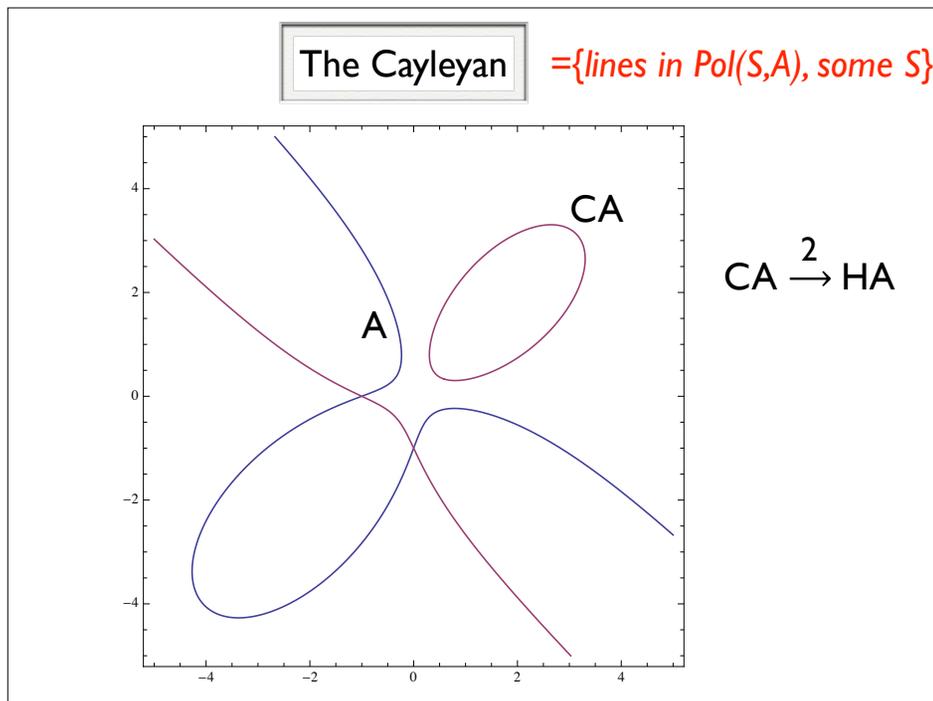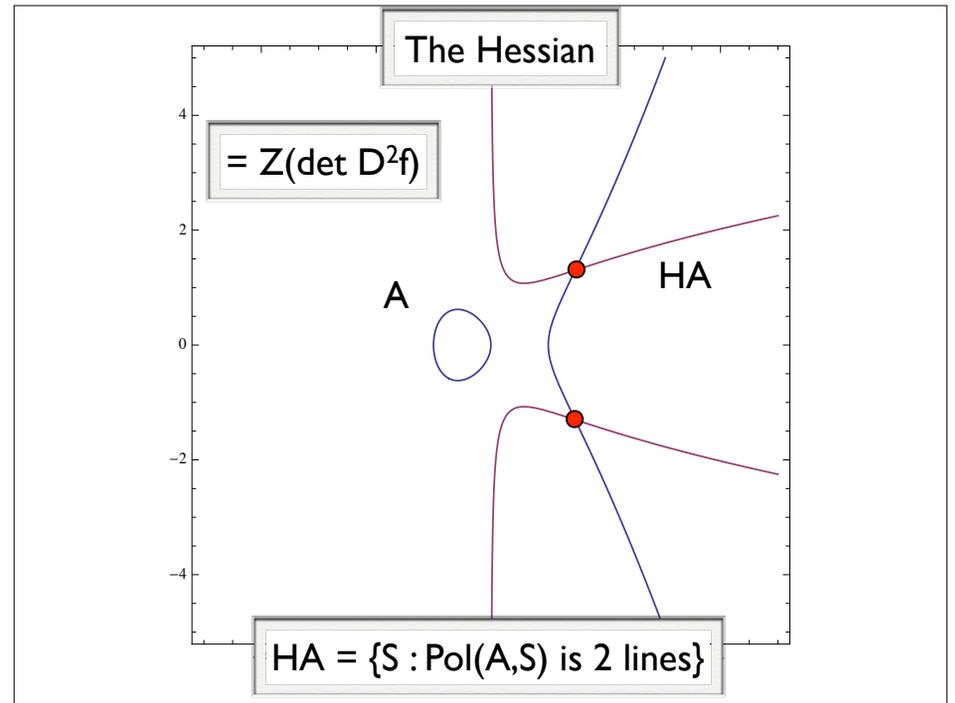INTENDED AS A SEQUEL

TO

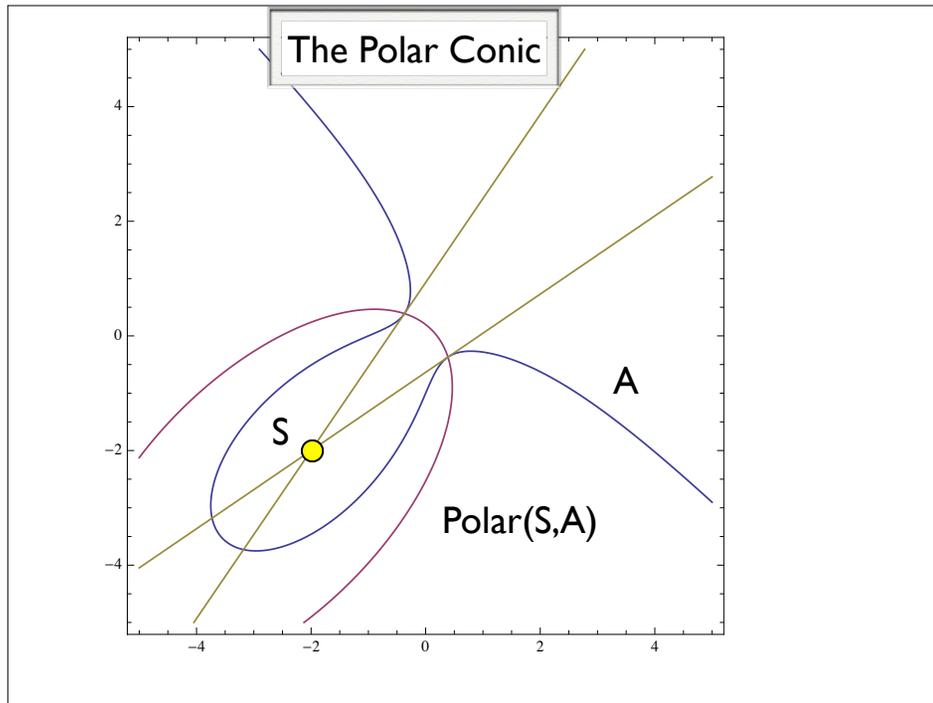A TREATISE ON CONIC SECTIONS.

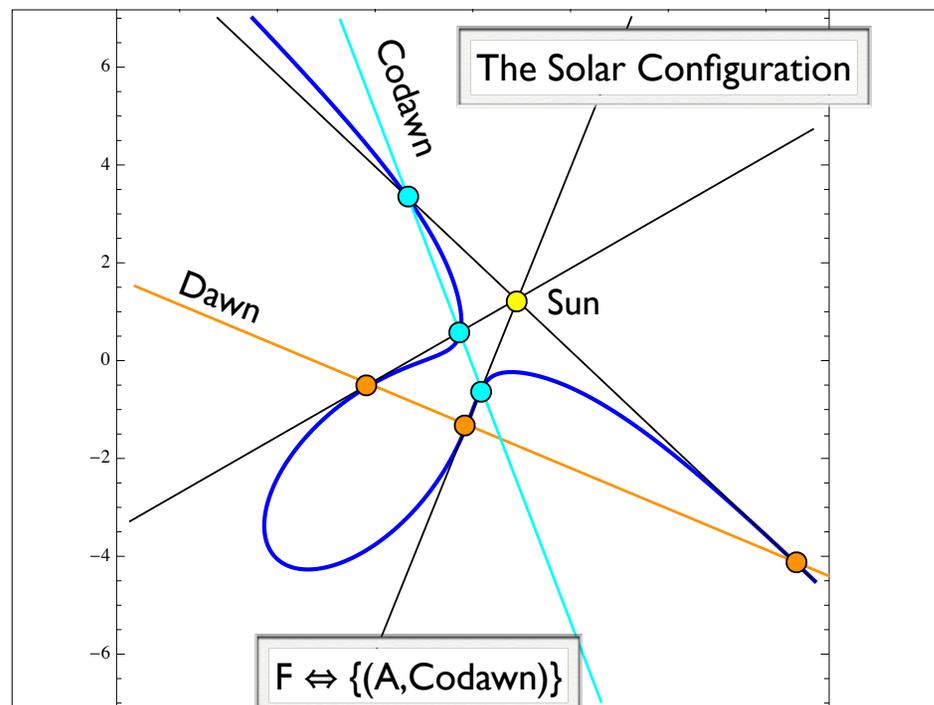BY

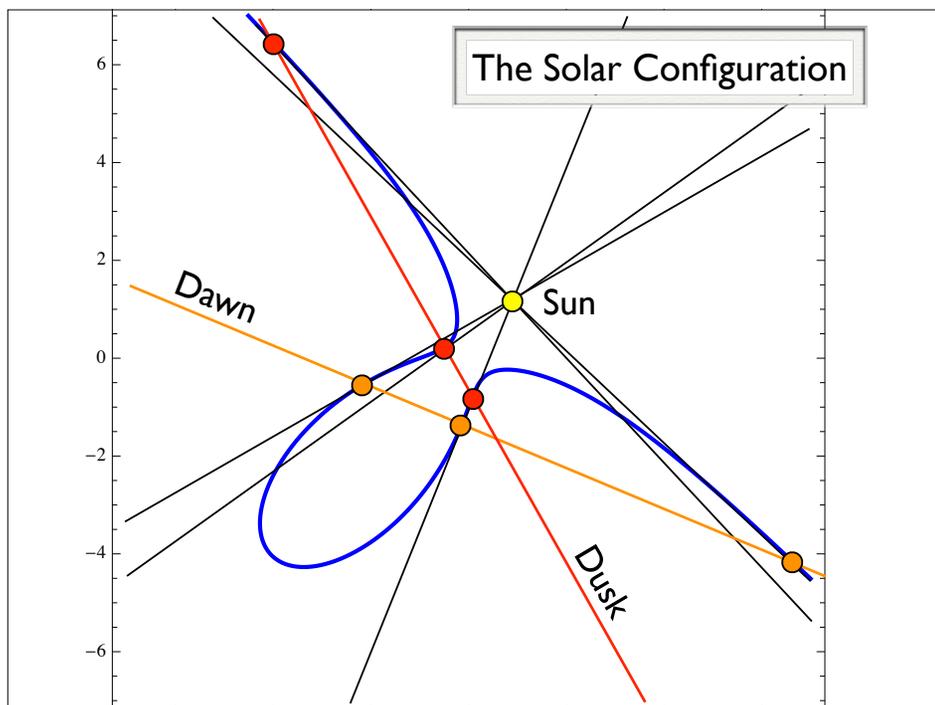GEORGE SALMON, D.D., D.C.L., LL.D., F.R.S.,
REGIUS PROFESSOR OF DIVINITY IN THE UNIVERSITY OF DUBLIN.

THIRD EDITION.

Dublin:
HODGES, FOSTER, AND FIGGIS, GRAFTON STREET,
BOOKSELLERS TO THE UNIVERSITY.
MDCCCLXXIX.

The Polar Conic

S

A

Polar(S,A)

The Hessian

$= Z(\det D^2 f)$

A

HA

HA = {S : Pol(A,S) is 2 lines}

The Cayleyan ={lines in Pol(S,A), some S}

CA

A

$CA \xrightarrow{2} HA$

The flex locus $F \subset \mathbb{M}_{1,3}$

is:

(A,P) in $\mathbb{M}_{1,3}$:
P = double(L) $\cap$ A, for some L in CA

$CA \rightarrow F \rightarrow \mathbb{M}_1$

Corollary: F is 2 dimensional.

The Solar Configuration

The Solar Configuration — Sun

The Solar Configuration — Dawn, Dusk, Sun

The Solar Configuration — Dawn, Codawn, Sun

$F \Leftrightarrow \{(A, Codawn)\}$
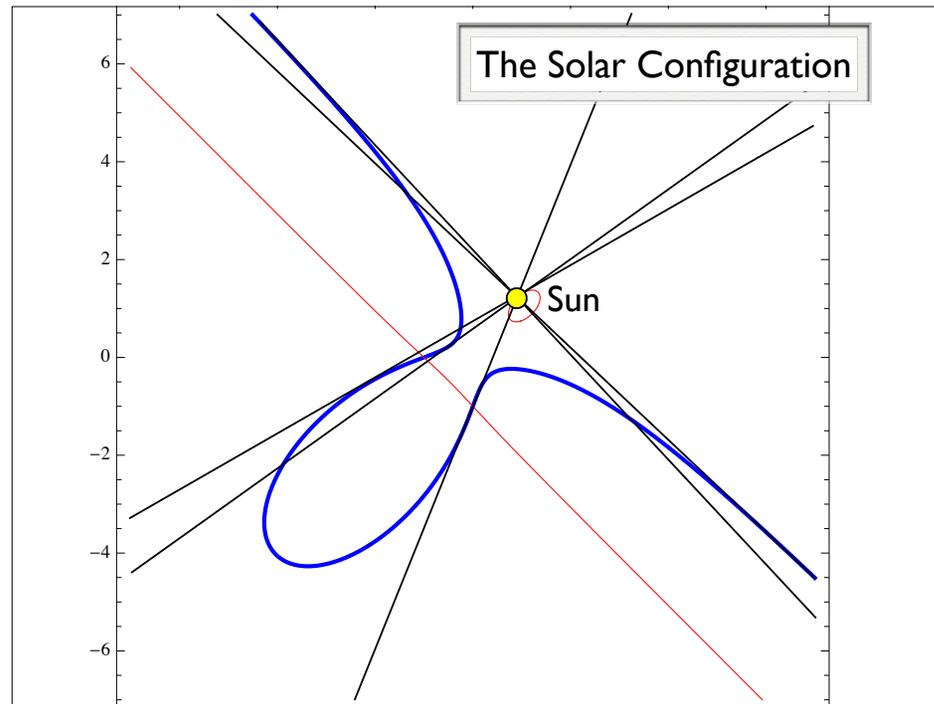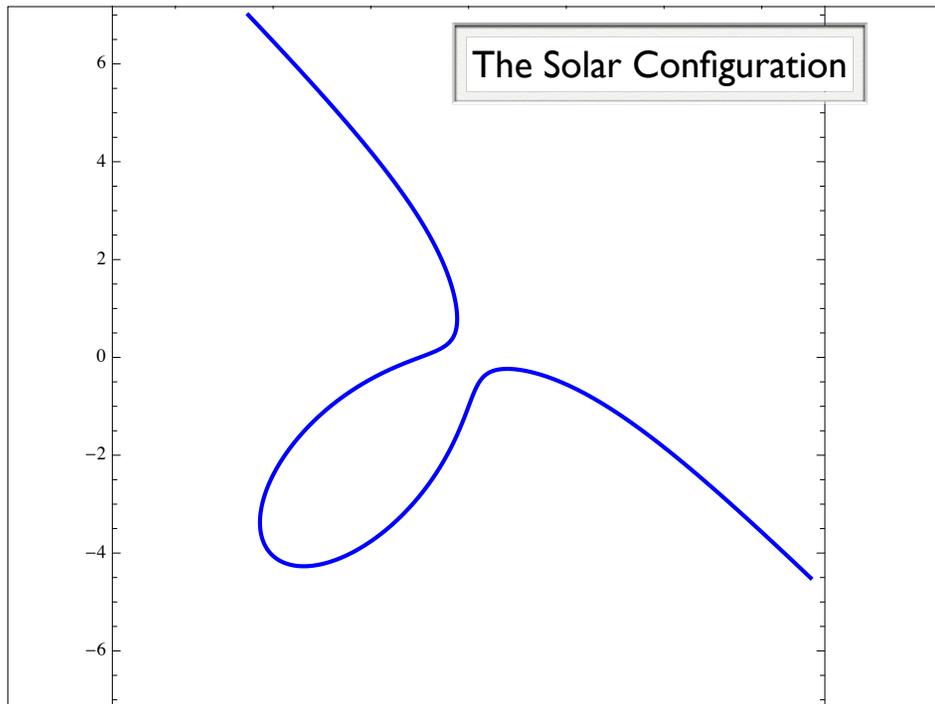
The gothic locus $\Omega G \subset \Omega\mathcal{M}_4$
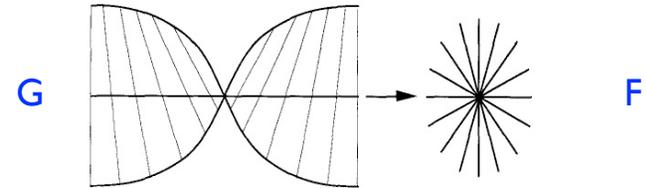
$\Omega G = \{(X,\omega)$ in $\Omega\mathcal{M}_4(2,2,2)$:

(i) $\exists$ J with A=X/J of genus 1;

(ii) ω is odd for J;

(iii) $\exists$ odd cubic map p: X $\to$B, genus 1;

(iv) p(Z(ω)) = one point.

Theorem: ΩG is an $SL_2(\mathbb{R})$ invariant 4-manifold.

From ΩG to F

(X, ω) in ΩG $\to$ $(A,q) = (X/J, \omega^2/J)$

$\to$ $(A,P$ = poles(q)) in F

G   F

Corollary: F is totally geodesic

Cathedral polygons

Wright
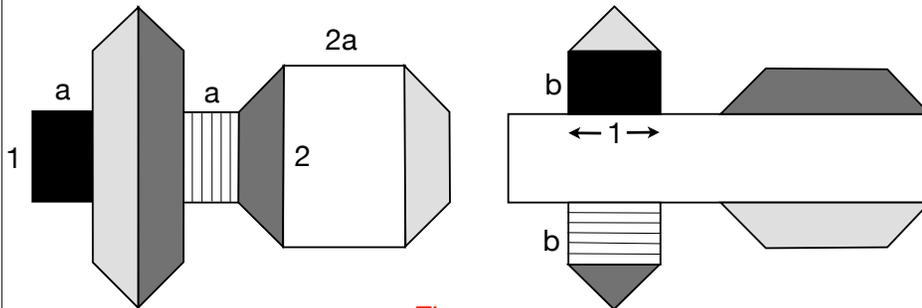
Theorem.
For every real quadratic field $K = \mathbb{Q}(\sqrt{d})$,
there exists a cathedral $P(a,b) \subset \mathbb{C}$ that
generates a Teichmüller curve V in $\mathcal{M}_4$.

Complement.
We have a new infinite series of
Teichmüller curves in the gothic locus $\mathbb{G} \subset \mathcal{M}_4$.

Known Teichmüller curves

$\mathcal{M}_6$

$\mathcal{M}_5$

?

$\mathcal{M}_4$ *MMW 2016*

$\mathcal{M}_3$ *M 2005 (Prym)*

$\mathcal{M}_2$ *Calta, M 2002 (Jacobian)*

$E_6,$
$E_7,$ *Veech 1989*
E_8 *Bouw-Möller 2006*

Q. What *are* the totally geodesic
surfaces in $\mathcal{M}_{1,3}$?

Why does F exist?