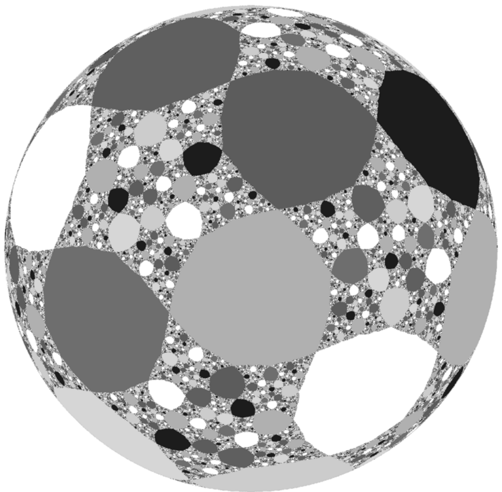
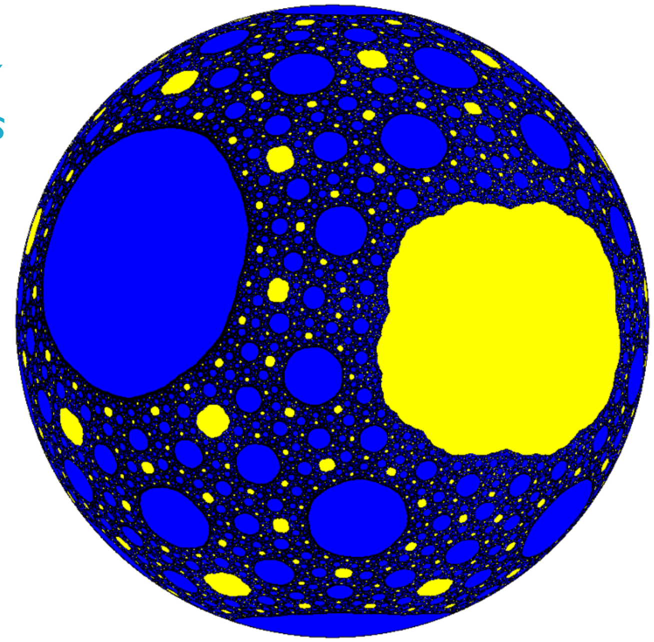
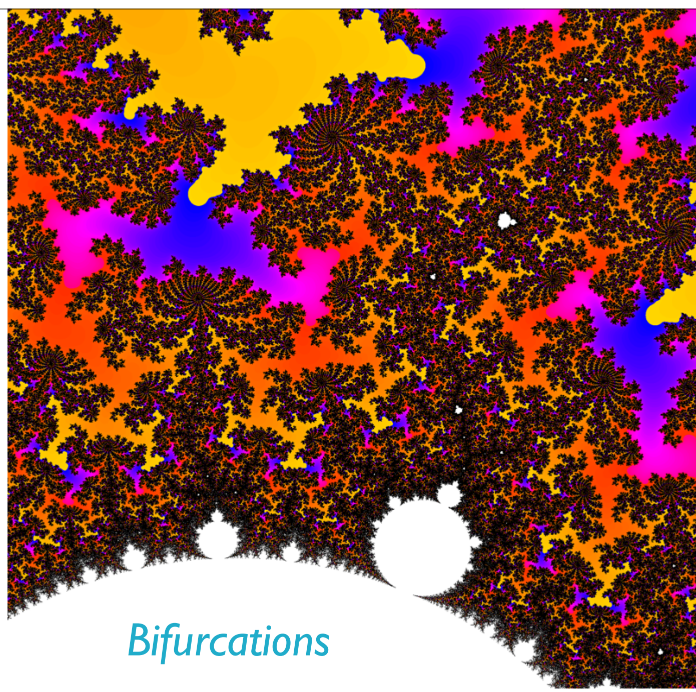
Solving polynomials: braids, rigidity and dynamics



Curtis T McMullen
Harvard University

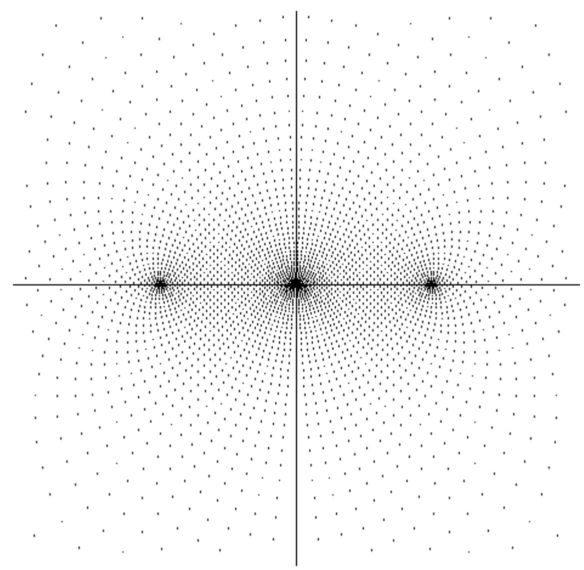
Complex dynamics





Bifurcations

Lattès examples



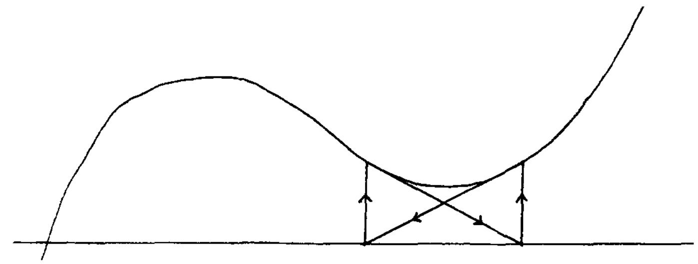
Rigidity

An algebraic family of rational maps
is either trivial, Lattès, or has bifurcations.

How to compute $\sqrt{2}$

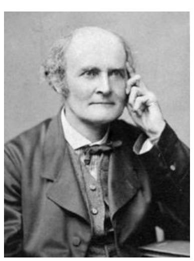
new guess for $\sqrt{2}$ = average of x + 2/x

1, 1.5, 1.41666, 1.414215, 1.4142136, ...



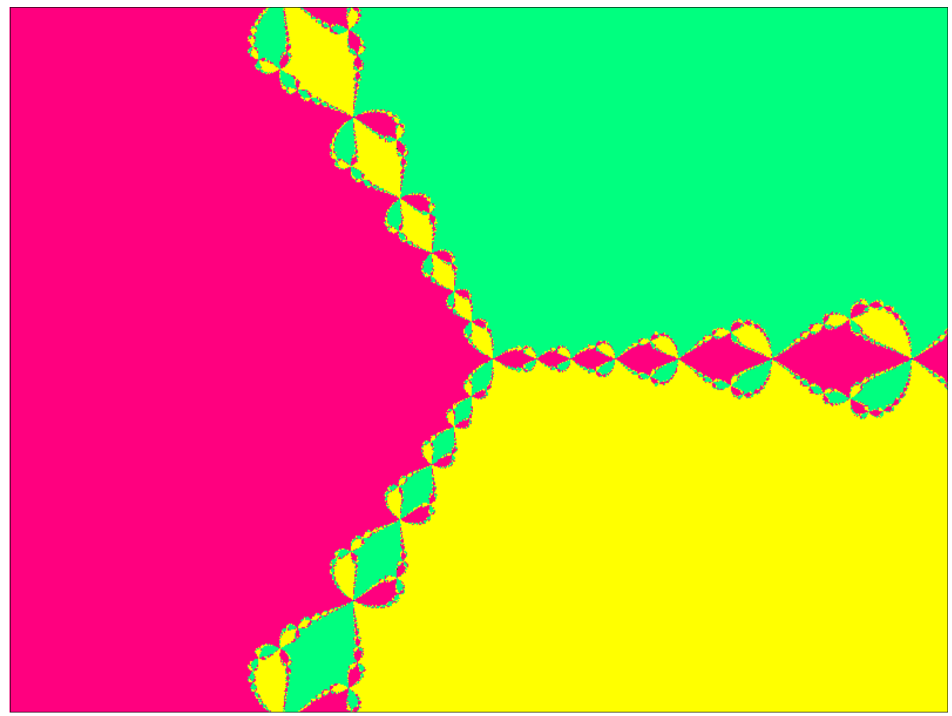
How to compute the cube root of 2

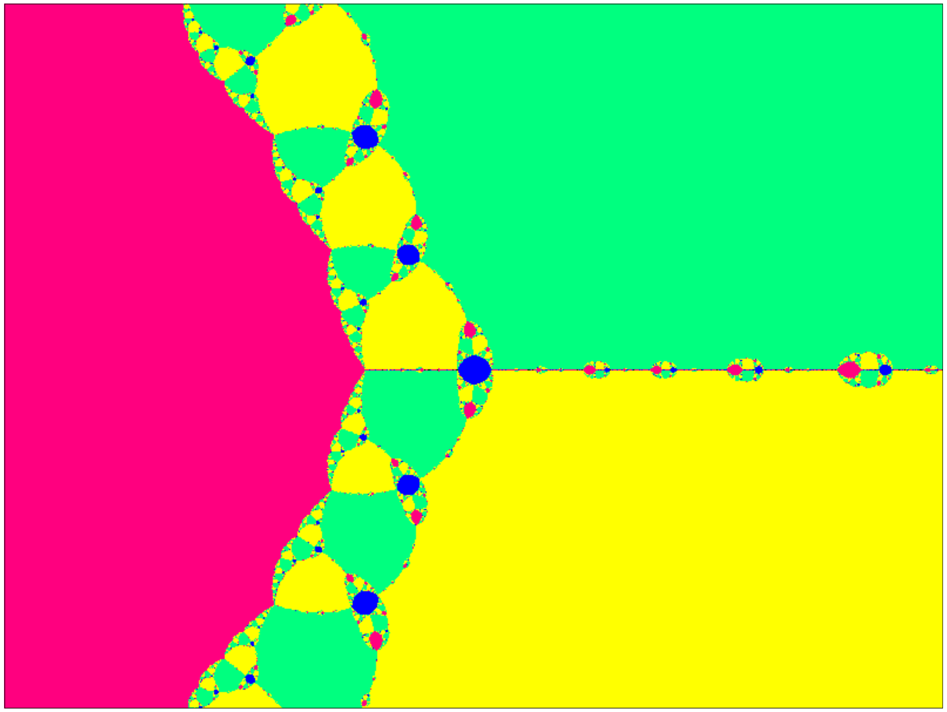
The problem is to determine the regions of the plane such that P, taken at pleasure anywhere within one region, we arrive ultimately at the point A, anywhere within another region we arrive at the point B, and so for the several points representing the root of the equation. The solution is easy and elegant for the case of a quadric equation; but the next succeeding case of a cubic equation appears to present considerable difficulty.

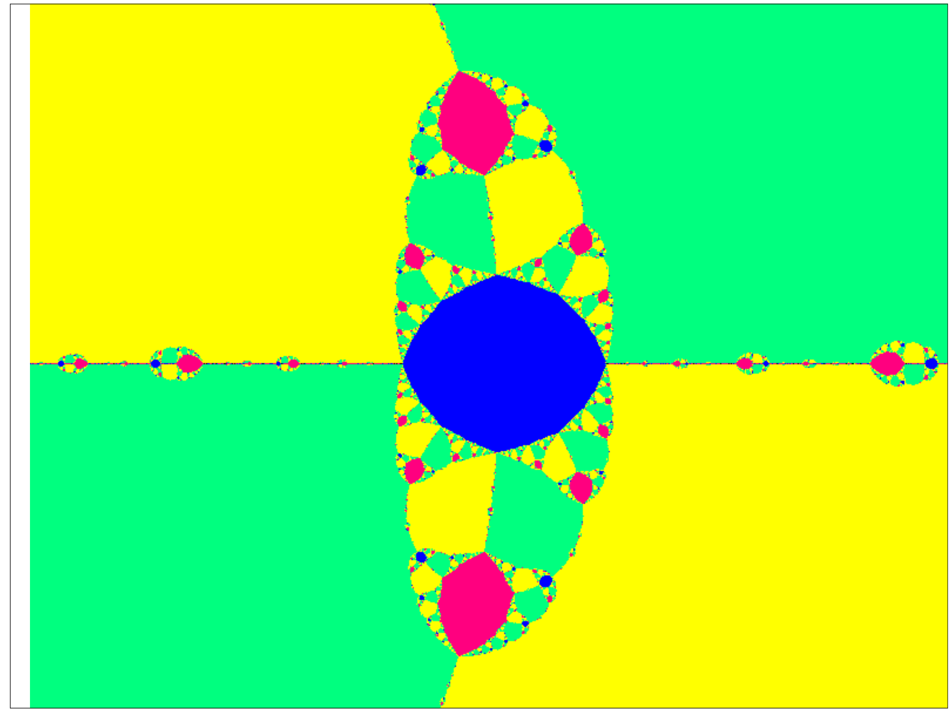


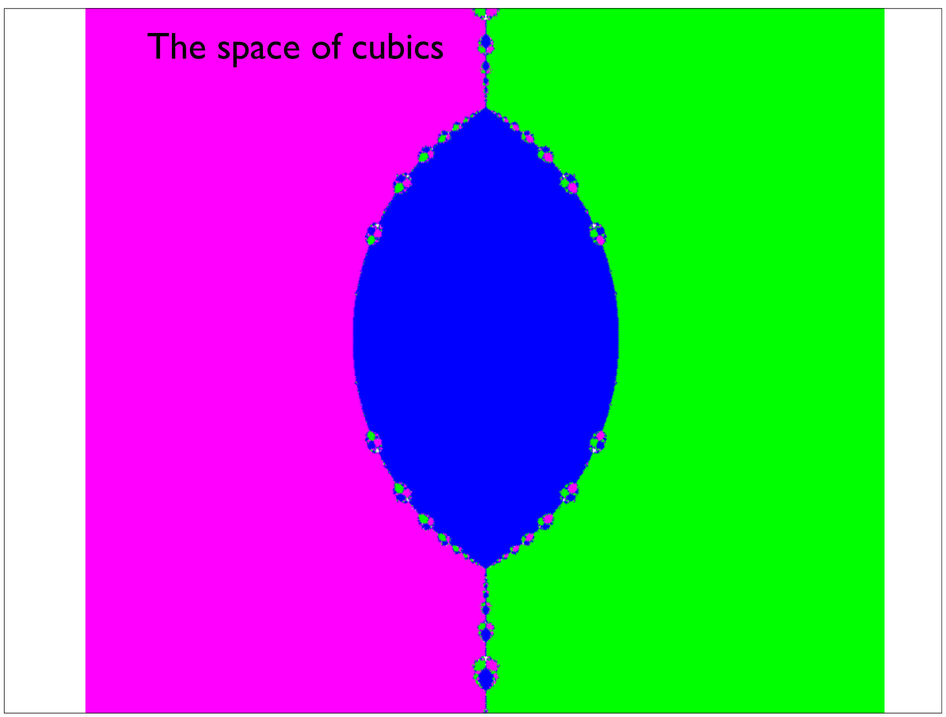
Sternberg, 2010

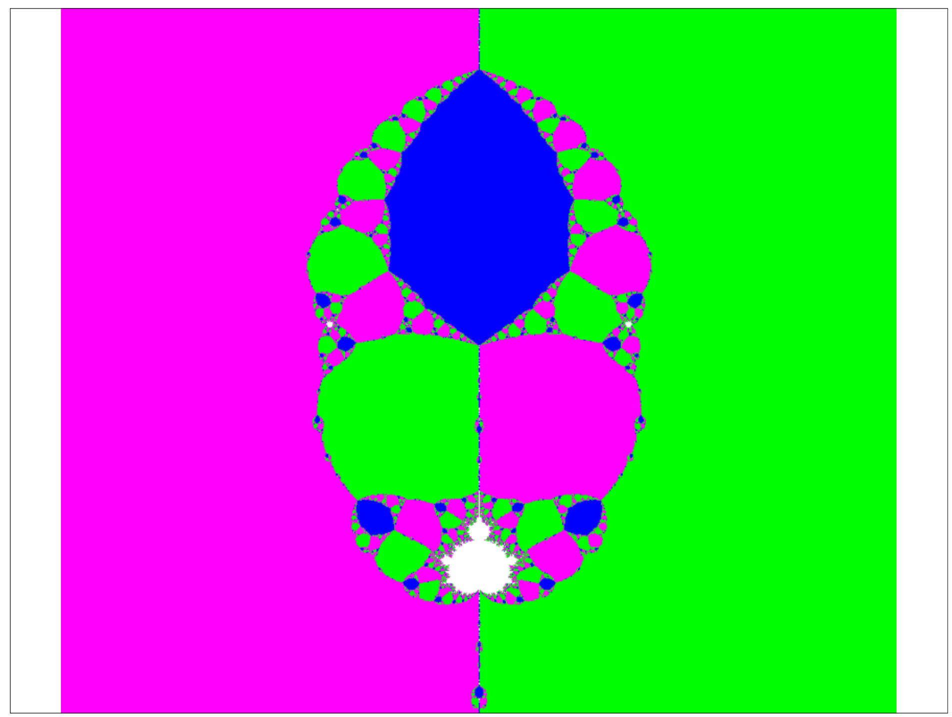
Arthur Cayley (August 16, 1821 - January 26, 1895)

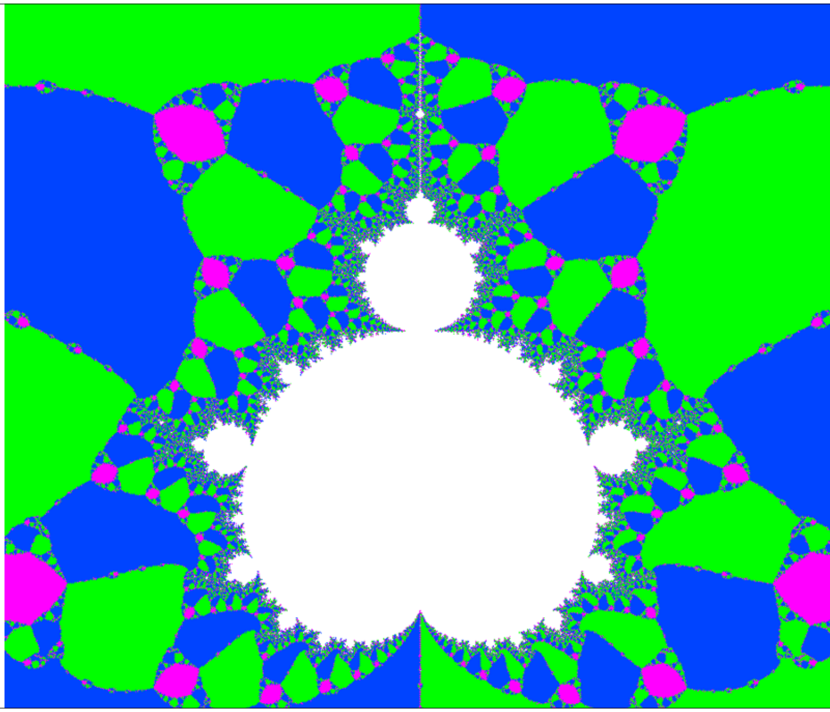







The space of cubics

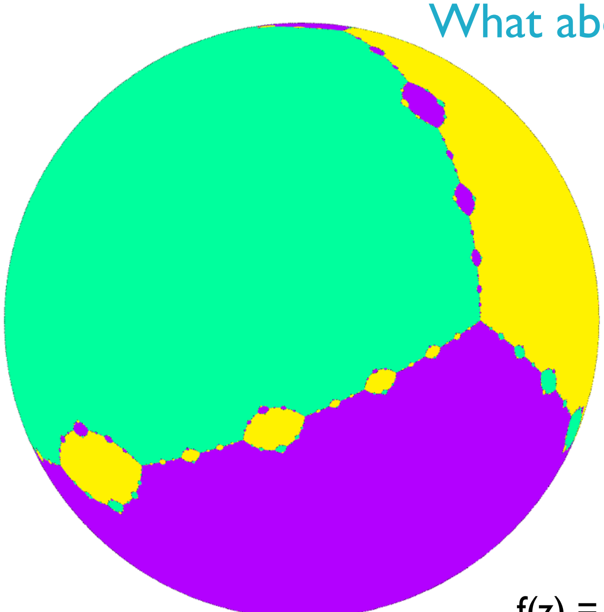




Unsolvability of the quartic

There is no purely iterative algorithm
to solve polynomials of degree 4 or more.

What about cubics?



S_3 symmetry

$f(z) = (z^4 + 2z)/(2z^3+1)$

How to solve cubics

II. *There exists a unique degree 4 superconvergent algorithm for cubics. If the cubic polynomial p is given by*

$$p(X) = X^3 + aX + b$$

then the algorithm is given by

$$T_p(X) = X - \frac{(X^3 + aX + b)(3aX^2 + 9bX - a^2)}{(3aX^4 + 18bX^3 - 6a^2X^2 - 6abX - 9b^2 - a^3)}\,.$$

= (Tate) Newton for

$$q(X) = \frac{p(X)}{(3aX^2 + 9bx - a^2)}$$

Braids

The braiding of the attractor in a stable family of rational maps is either

- reducible,
- finite, or
- it fixes a point of the attractor.

Location of failures

Every algorithm fails somewhere along this loop in the space of degree 4 polynomials

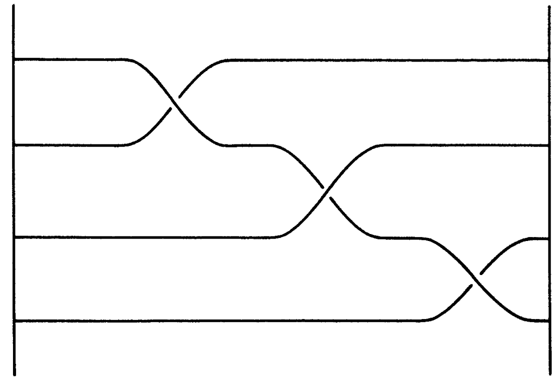
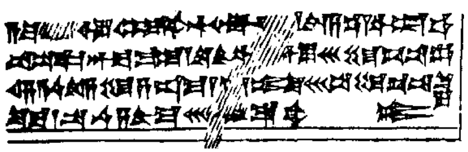


Fig. 2.1. This braid does not arise for rational maps

Solving polynomials through the ages

Various authors

Solving the quadratic, circa 2000 BC



Solving the cubic, circa 1500 AD

REGULA.

Deducito tertiam partem numeri rerum ad cubum, cui addes quadratum dimidij numeri æquationis, & totius accipe radicem, scilicet quadratam, quam seminabis, uniq; dimidium numeri quod iam in se duxeras, adijcies, ab altera dimidium idem minues, habebisq; Binomium cum sua Apotome, inde detracta ℞ cubica Apotomæ ex ℞ cubica sui Binomij, residuũ quod ex hoc relinquitur, est rei estimatio.

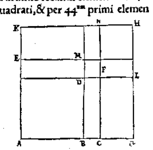
Exemplum. cubus & 6 positiones, æquantur 20, ducito 2, tertiam partem 6, ad cubum, fit 8, duc 10 dimidium numeri in se, fit 100, iunge 100 & 8, fit 108, accipe radicem quæ est ℞ 108, & eam geminabis, alteri addes 10, dimidium numeri, ab altero minues tantundem, habebis Binomiũ ℞ 108 p: 10, & Apotomen ℞ 108 m: 10, horum accipe ℞ cub & minue illam quę est Apotomæ, ab ea quæ est Binomij, habebis rei æstimationem, ℞ v: cub: ℞ 108 p: 10 m: ℞ v: cubica ℞ 108 m: 10.

cub⁹ p: 6 reb⁹ æq̃lis 20
2 20
8 —— 10
108
℞ 108 p: 10
℞ 108 m: 10
℞ v: cu. ℞ 108 p: 10
m: ℞ v: cu. ℞ 108 m: 10

Solving the quartic, circa 1500 AD

DEMONSTRATIO.

Sit quadratum A F, diuisum in duo quadrata A D & D F, & duo supplementa D C & D E, & uelim addere gnomonem K F G circũcirca, ut remaneat quadratum totum A H, dico quod talis gnomo, constabit ex duplo G C additæ lineæ, in C A, cum quadrato G C, nam F G constat ex G C in C F, ex diffinitione data in initio secundi elementorum, et C F est æqualis C A, ex diffinitione quadrati, & per 44 primi elementorum, K F est æqualis F G, igitur duæ superficies G F & F K, constant ex G C, in duplum C A, & quadratũ G C est F H, ex corrolario quartæ secundi elementorũ, igitur patet propositum, si igitur A D sit 1 q̃d q̃d & C D ac D E, 3 quadrata, & D F 9, erunt B A 1 quadratum, & B C 3 necessario, cum igitur uoluerimus addere q̃drata aliqua, ad D C & D E, & fuerint C L & K M, erit ad cõplendum quadratum totum necessaria superficies L N M, quæ ut demonstratum est, constat ex quadrato G C numeri quadratorum dimidiati,



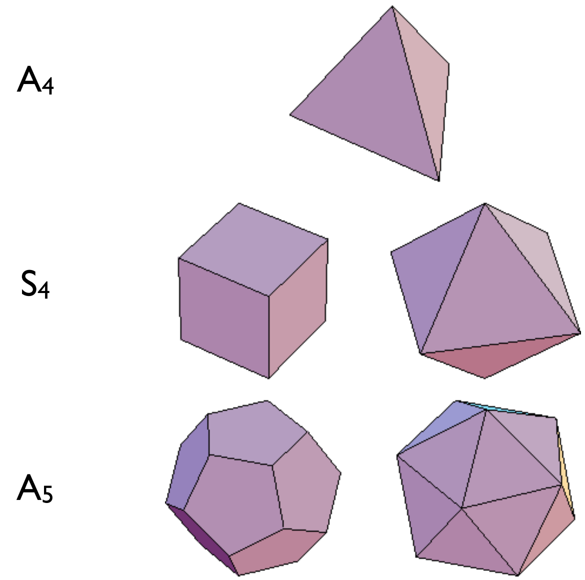
Insolvability of the quintic 1824

Towers of algorithms

A field extension can be computed by a tower of purely iterative algorithms iff its Galois group is within A_5 of solvable.

Solvability of the quintic

The quintic can be solved by a
tower of algorithms, but the sextic cannot.



Rational maps with A_5 symmetry

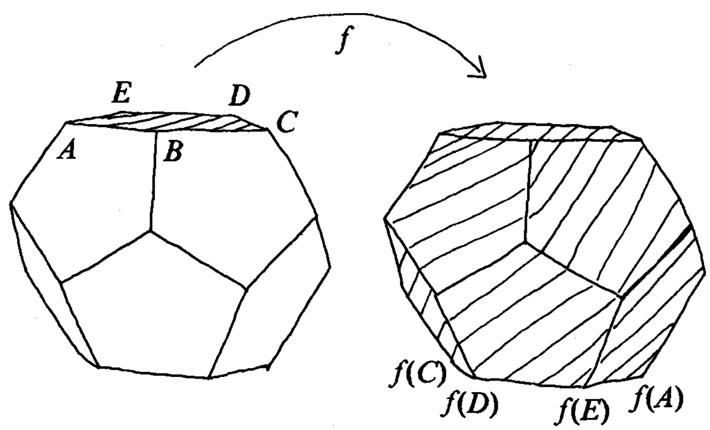


Fig. 4. Geometric construction of a rational map.

Vertices, face centers, edge midpoints

$$f = x^{11}y + 11x^6y^6 - xy^{11}$$

$$H = -x^{20} - y^{20} + 228(x^{15}y^5 - x^5y^{15}) - 494x^{10}y^{10}$$

$$T = x^{30} + y^{30} + 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10} + y^{10}y^{20}).$$

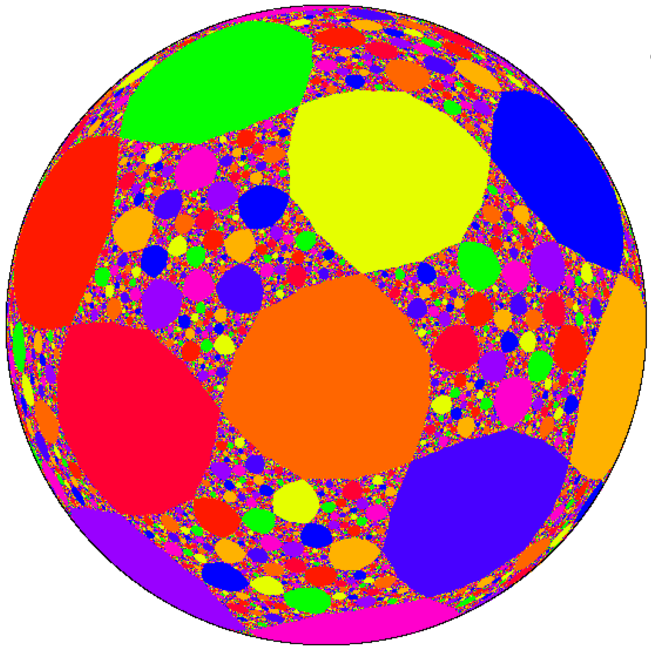
PROPOSITION 5.3. *There are exactly four rational maps of degree <31 which commute with the icosahedral group. These four maps, of degree* 1, 11, 19 *and* 29 *respectively, are:*

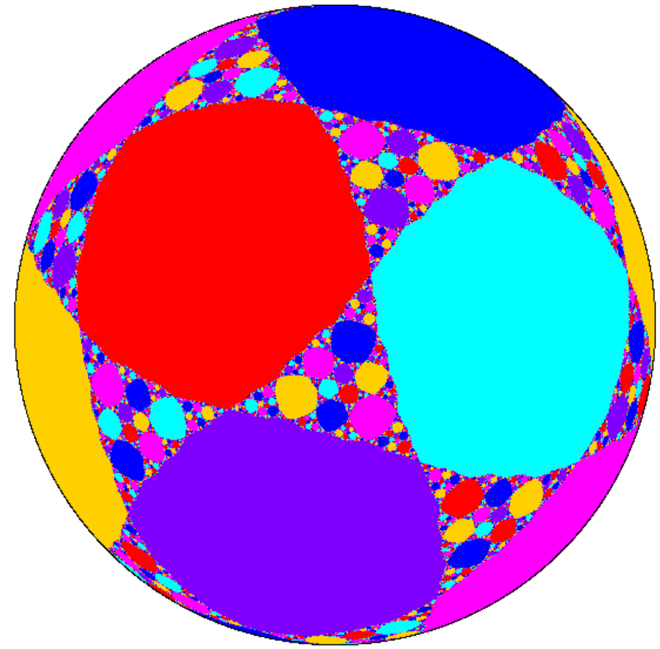
$$f_1(z) = z$$

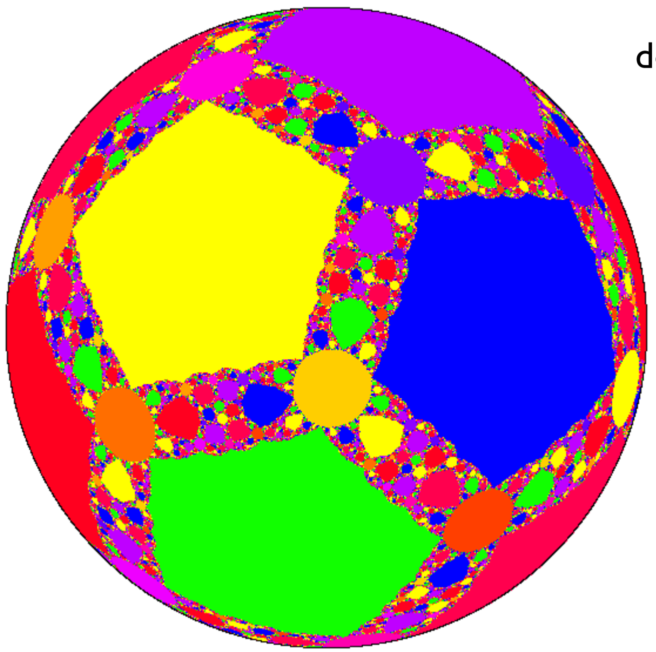
$$f_{11}(z) = \frac{z^{11} + 66z^6 - 11z}{-11z^{10} - 66z^5 + 1}$$

$$f_{19}(z) = \frac{-57z^{15} + 247z^{10} + 171z^5 + 1}{-z^{19} + 171z^{14} - 247z^9 - 57z^4}$$

$$f_{29}(z) = \frac{87z^{25} - 3335z^{20} - 6670z^{10} - 435z^5 + 1}{-z^{29} - 435z^{24} + 6670z^{19} + 3335z^9 + 87z^4}.$$







Degree 6

z → algorithm → f(z) : multivalued, algebraic

Can polynomials of degree 6 be solved using algebraic functions of just one variable?

Hilbert's 13th problem