Dynamics and algebraic integers: Perspectives on Thurston's last theorem

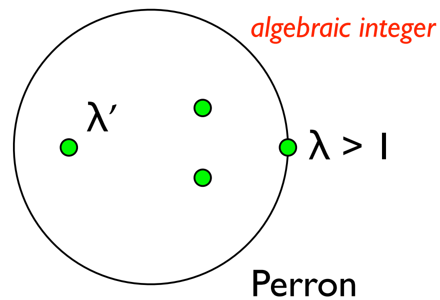
Curtis McMullen
Harvard University

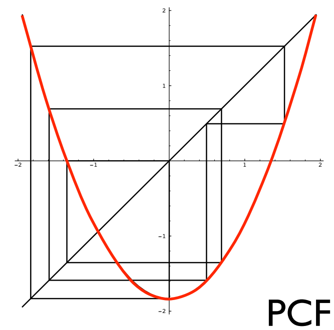
Theorem (Thurston, *Entropy in dimension one,* 2014)

There exists a post-critically finite map $f : I \to I$ with entropy $h(f) = \log(\lambda)$

$\Leftrightarrow$

λ is a weak Perron number.





Theorem (Thurston, *Entropy in dimension one,* 2014)

For any Perron number λ, there exists an $f : I \to I$ with $h(f) = \log(\lambda)$

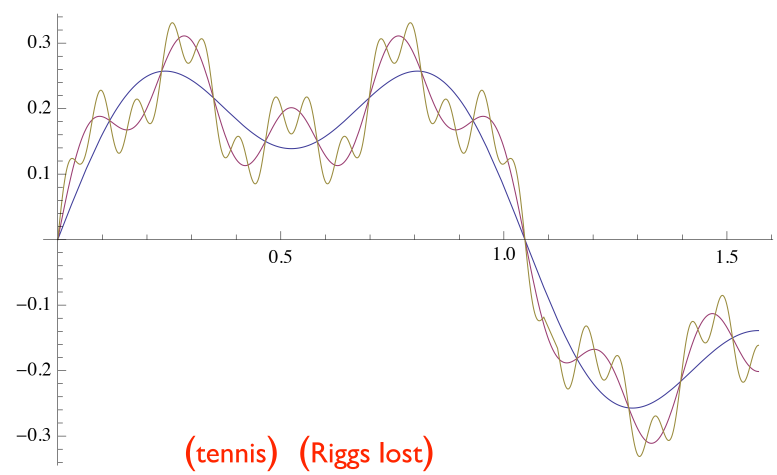
Thesis: *This result is an instance of the h-principle.*

Given enough flexibility, anything that is not obstructed will actually occur.

Defn. *``Every continuous section of a suitable partial differential relation in a jet bundle is homotopic to a holonomic section.''*

Examples of the h-principle

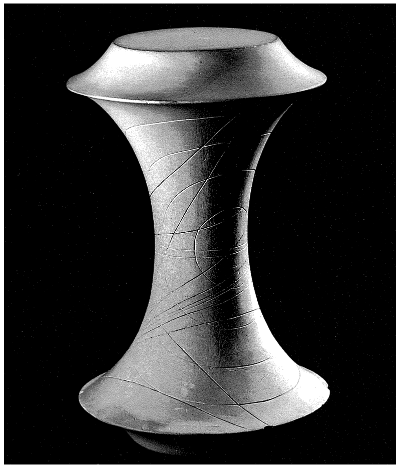
There exists a continuous, nowhere differentiable function. (Weierstrass)



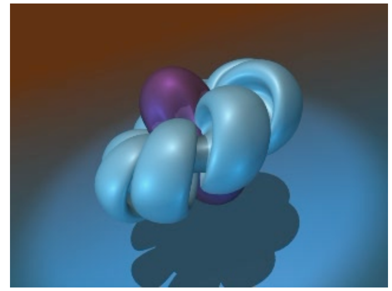
(tennis) (Riggs lost)

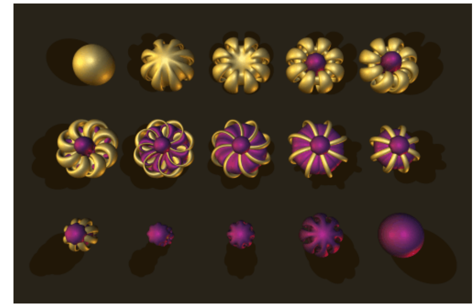
A closed hyperbolic surface embeds isometrically (C^1) into $\mathbb{R}^3$.

(Nash-Kuiper)



The sphere can be turned inside-out. (Smale)





(Thurston)

There exists a path-isometry $f : S^2 \to \mathbb{R}^2$.

(Length of γ = length of $f \circ \gamma$.) (Gromov)

$\xrightarrow{f}$

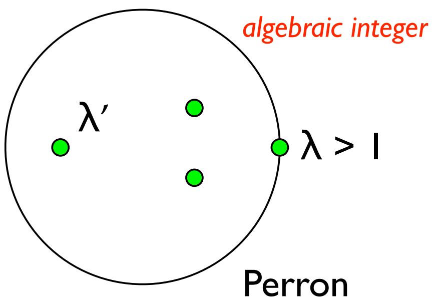
A manifold admits a codimension 1 foliation $\Leftrightarrow$ $\chi(M) = 0$ (Thurston)

(Lawson, Haefliger, Bott)

There exists a Perron-Frobenius matrix
P in $M_n(\mathbb{Z})$ with spectral radius $\rho(P) = \lambda$ (Lind)

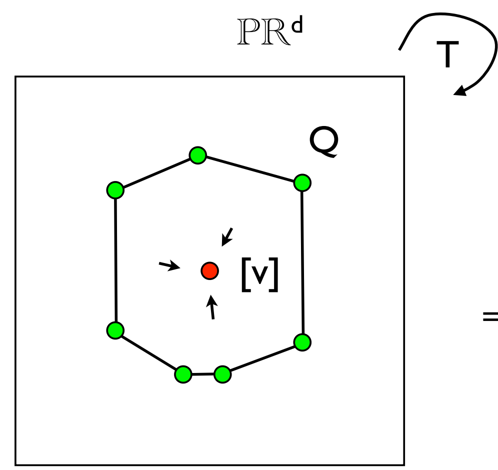
$\Leftrightarrow$

λ is a Perron number.



Perron-Frobenius means
$P_{ij} \geq 0$
and $P^k_{ij} > 0$ for some k.

Constructing P from λ $\deg(\lambda) = d$



T in $M_d(\mathbb{Z})$
$Tv = \lambda v$

find convex n-gon Q,
$T(Q) \subset Q$

$\Rightarrow$ PF matrix P in $M_n(\mathbb{R})$,
$\rho(P) = \lambda$.

Even for cubic λ, we may be forced to take $n \gg 0$. *(M, Lind)*

Theorem (Thurston)

There exists a post-critically finite
map $f : I \rightarrow I$ with entropy $h(f) = \log(\lambda)$

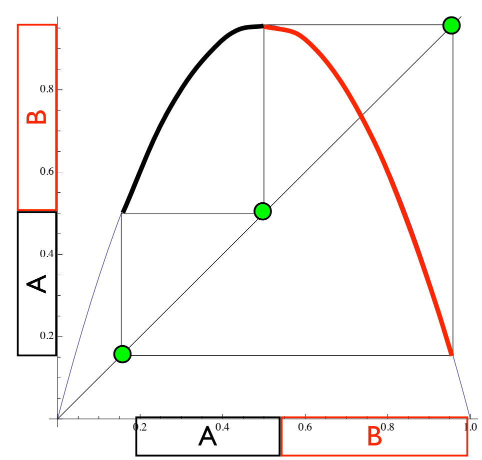
$\Leftrightarrow$

λ is a (weak) Perron number.

Why is entropy is Perron? *Markov partitions*

=unifies=
PCF multimodal maps
pseudo-Anosov maps
automorphisms of free groups

Golden mean example



A ⇄ B ↺

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\lambda = \rho(P) = \frac{1+\sqrt{5}}{2}$$

For any pcf f, h(f) = log(Perron)

Constructing f from λ

For any Perron number λ,
there exists an $f : I \to I$ with $h(f) = \log(\lambda)$

{Perron numbers λ} = $\bigcup M_k$

M_k = {λ arising from f with k laps}

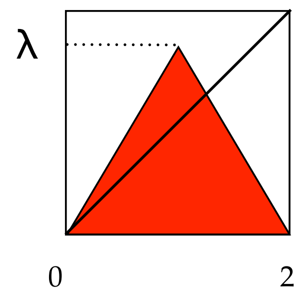
STABLE

At any stage in the construction one can increase # laps or size of post-critical set.

[*Proof.* First step is Lind's theorem.]

Quadratic Entropy Problem *UNSTABLE*

Describe M_2 = {λ arising from quadratic f}.



λ is in M_2 ⇔ 0 has finite orbit under $T_\lambda(x) = \lambda(1 - |1-x|)$

Q. Does there exist an algorithm to test if λ *is not in* M_2?

Known: Pisots ⊂ M_2

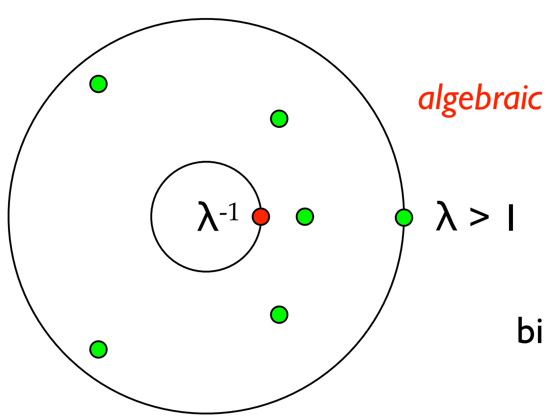
Unknown (e.g.): Is Lehmer's number in M_2?

Surface Entropy Conjecture *STABLE (open)*

There exists a pseudo-Anosov map

$f : S \to S$ with $h(f) = \log(\lambda)$ (Fried 1985, Thurston)

⇔ *λ is a biPerron number.*



algebraic unit

biPerron

Minimum entropy problem for surfaces

$\log(\delta_g)$ = min {h(f) : f pseudo-Anosov on a surface of genus g}

= length of shortest geodesic on $\mathcal{M}_g$.

Example: δ_1 = root of t^2-3t+1 $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

δ_2 = root of $t^4-t^3-t^2-t+1$ *(Cho-Ham)* *(Lanneau–Thiffeault)*

Problem: Determine δ_g for all g.

Minimum entropy problem for surfaces

Known: $\delta_g = 1 + O(1/g)$ (Penner, 1991)

Question: Does $\lim (\delta_g)^g$ exist? (M, 2000)

> Conjecture: $\lim (\delta_g)^g = \delta_1$
>
> Theorem: $\limsup (\delta_g)^g \leq \delta_1$
>
> (E. Hironaka, 2010)

(Aaber-Dunfield, Kin, ...)

Q. Where do f with h(f) = O(1/g) come from?

A. Fibered 3-manifolds.

M = hyperbolic 3-manifold fibering over S^1, $b_1(M) > 1$

⇒ M fibers in infinitely many ways

⇒ infinitely many $f : S \to S$ whose mapping torus is M.

- Among these are f with $h(f) = O(1/g)$ (M, 2000)
- All such f arise from finitely many M.

(Farb-Leininger-Margalit, 2011)

A manifold M from δ_1

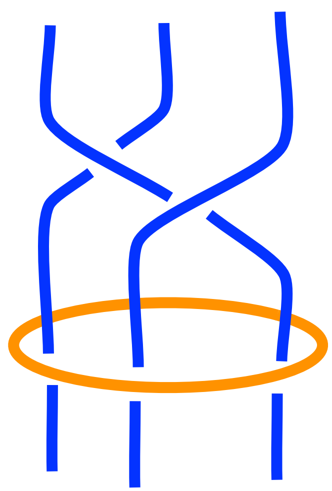
(E. Hironaka, 2010)

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

$$= \mathrm{Mod}_1 \cong \mathrm{Mod}_{0,4}$$

⇕

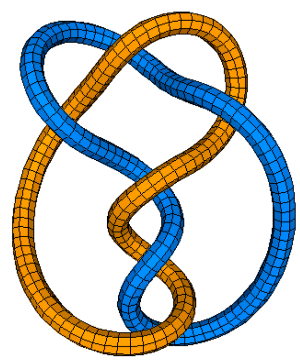
Braids on 3 strands



$M^3 = S^3$-L fibers over S^1

Entropy of other fibrations $M^3 \to S^1$

$M^3 = S^3$-L



$H_1(M)$

Teichmüller polynomial

	-1	
1	-1	1
	-1	

t^2-3t+1

$$L_{2g}(t) = t^{2g} - t^g(t+1+t^{-1}) + 1$$

Teichmüller polynomial: specializations

$L_4(t) = t^4 - t^3 - t^2 - t + 1$

$\Rightarrow$ minimal genus 2 example

$L_{2g}(t)$ $\Rightarrow$ genus g examples with $\delta(f)^g \sim \delta_1$, in agreement with conjecture

$$L_{2g}(t) = t^{2g} - X^g(X+1+X^{-1}) + 1 \approx t^{2g} - 3t^g + 1$$

Conclusion: $\limsup (\delta_g)^g \leq \delta_1$

Remaining problem: lower bounds for δ_g.

Is there a more tractable, algebraic problem?

$[f : S \to S] \Rightarrow$

train track map $[F : \tau \to \tau] \Rightarrow$

Perron-Frobenius P, $\delta(f) = \rho(P)$.

Symplectic structure on $\mathbb{P}ML$ $\Rightarrow$ P is reciprocal.

(eigenvalues invariant under $t \to 1/t$*)*

Q. Can we give a lower bound on $\rho(P)$?

Minimum entropy problem for graphs

Theorem *(M, 2013)*

Min $\rho(P)$: P in $M_{2g}(\mathbb{Z})$ is reciprocal and Perron-Frobenius

=

Largest root of

$$L_{2g}(t) = t^{2g} - t^g(t+1+t^{-1}) + 1$$

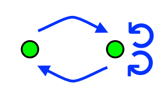
Corollary

$\rho(P)^g \geq \delta_1$ for all P and g.

Scheme of the proof

P in $M_n(\mathbb{Z})$, Perron-Frobenius

$\Rightarrow$ metrized directed graph (Γ, l)



#(closed loops $\leq T) \asymp \rho(P)^T$

Optimal metric on $\Gamma \Rightarrow$ invariant $\lambda(\Gamma)$

$\rho(P)^n \geq \lambda(\Gamma)$

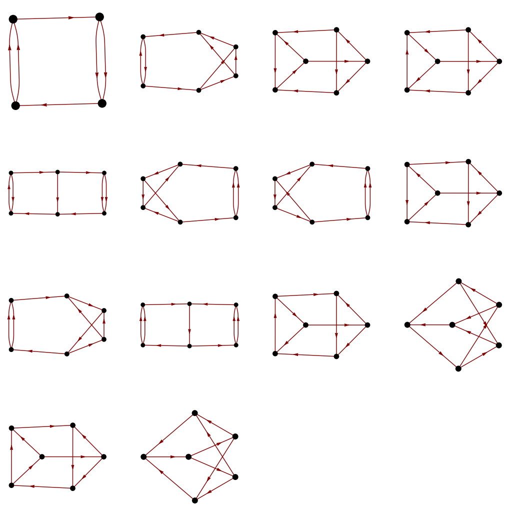
- *Fact:* $\{\Gamma : \lambda(\Gamma) < M\}$ *is finite*

Case at hand, take M=8.

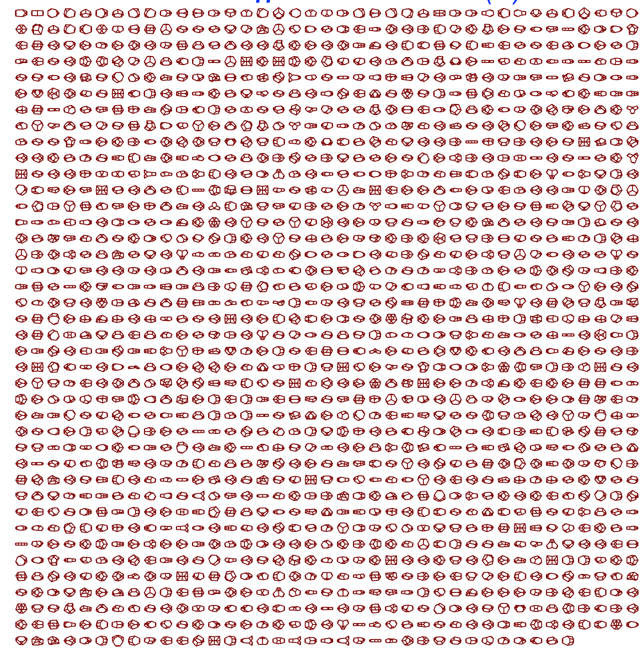
$(\delta_1)^2$ = (golden ratio)^4 =6.854...

Scheme of the proof

Analyze each Γ : λ(Γ) < 8. *(2009)*



1556 different Γ with λ(Γ)≤6



Scheme of the proof

(2013)

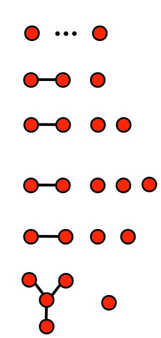
Γ ⇒ curve complex G

V(G) = {simple loops C in Γ}
E(G) = {disjoint (C,D)}

- G carries key info of Γ
- New invariant: λ(Γ) ≥ λ(G)
- {G : λ(G) < 8} = 11

(RAAGs, Cartier-Foata 1968)
(Birman, 2011)

List of G with λ(G) < 8

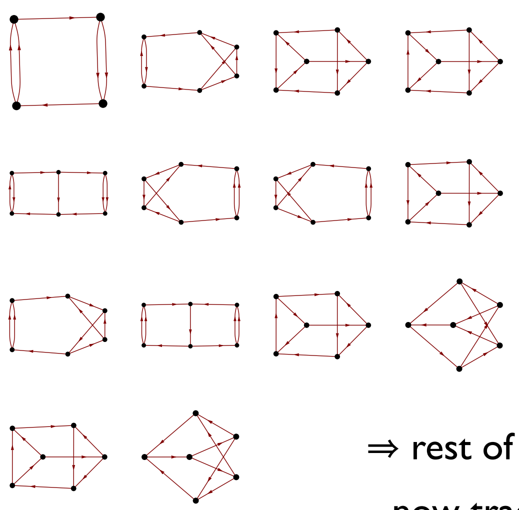


G	λ(G)
nA_1	$2 \leq n \leq 7$
A_2*	4
A_2**	5.82... = $3+2\sqrt{2}$
A_2***	7.46... = $4+2\sqrt{3}$
A_3*	5.82... = $3+2\sqrt{2}$
Y*	7.46... = $4+2\sqrt{3}$

G	$2A_1$	$3A_1$	$4A_1$	$5A_1$	$6A_1$	$7A_1$	A_2^*	A_2^{**}	A_2^{***}	A_3^*	Y^*
$\#\Gamma$	1	2	14	119	1556	26286	1	5	42	5	42

Table 2. Number of trivalent Γ with a given curve complex G.

Example: These Γ all have $G = 4A_1$ = ● ● ● ●



⇒ rest of proof is now tractable.

Concluding Remarks

M : Teichmüller polynomial
Γ : Perron polynomial
G : Clique polynomial

thermodynamic formalism
convexity of pressure

Problem: Determine δ_g for all g?

SEMISTABLE

EPILOGUE
SPOKEN BY PROSPERO
Now my charms are all o'erthrown,

* * *

Gentle breath of yours my sails
Must fill, or else my project fails,
Which was to please.

Better for more people to be involved...
it's a fun and interesting topic.