Algebraic Curves and Complex Dynamics

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Preliminaries

Algebraic Curves over Number Field

K number field $/ \mathbb{Q}$

X smooth curve $/ K$.

<u>Thm</u> $g(X) \geq 2 \Rightarrow \# X(K)$ finite.

ex. $x^n + y^n = 1$, $n = 4$ $\quad g = 3$. $K = \mathbb{Q}$

Proof uses ℓ-adic Galois representations

$$\mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}_\ell).$$

Algebraic Curves in $\mathcal{M}_g$ $\quad g \geq 1$

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"', "\text{proper}"] & \text{surface} \\ V & \text{curve} \end{tikzcd}$$

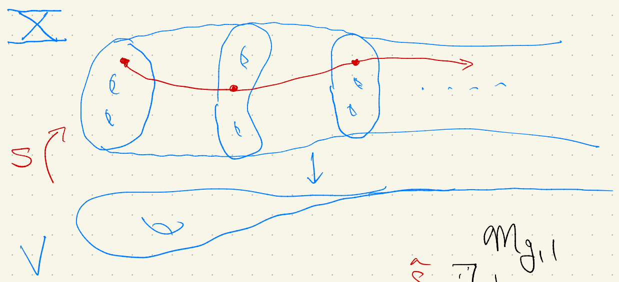
$$V \longrightarrow \mathcal{M}_g$$

= families of Riemann surfaces

<u>Ex.</u> $X_t : y^2 = (x^5 - 1)(x - t) \in \mathcal{M}_2$

$$\begin{tikzcd} X_t \arrow[d] \\ t \in V = \mathbb{C} - (n^{th} \text{ roots of } 1) \end{tikzcd}$$

Sections



$$\begin{tikzcd} & \mathcal{M}_{g,1} \arrow[d] \\ V \arrow[ur, dashed, "\hat{S}"] \arrow[r] & \mathcal{M}_g \end{tikzcd}$$

<u>Thm</u> If $\mathcal{X} \to V$ has ∞ many sections, then $\mathcal{X}$ is <u>trivial</u> or $g = 1$.

(Trivial – $\mathcal{X} = V \times X$; $\underset{\text{const.}}{V \to \mathcal{M}_g}$)

Proof: Schwarz lemma + rigidity

I. Algebraic Families of Rational Maps

$$\begin{tikzcd} V\times\mathbb{P}^1 \circlearrowleft F \arrow[d] \\ V \end{tikzcd}$$

preserves fibers.

curve

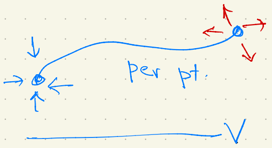
$V \to MRat_d$

Ex. $z^2+c : \mathbb{C}\times\mathbb{P}^1 \circlearrowleft F$

$$\begin{tikzcd} \mathbb{C}\times\mathbb{P}^1 \arrow[d, "c"] \\ V=\mathbb{C} \end{tikzcd}$$

Thm (M). A dynamically stable family of rational maps F is either ~~trivial~~ or "genus one". (crux)

Stable = no bifurcations



genus one

elliptic curves

$$\begin{tikzcd} E \circlearrowleft \tilde{F}(x)=nx \arrow[d] \arrow[rr, dotted, "x\sim -x"'] & & E/\sim \;\simeq V\times\mathbb{P}^1 \circlearrowleft F \arrow[d] \\ V & & V \end{tikzcd}$$

(Lattès)

Methods

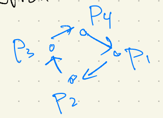
Study sections:

$$\begin{tikzcd} V\times\mathbb{P}^1 \circlearrowleft F \arrow[d] \\ V \arrow[u, bend left, "s"] \end{tikzcd}$$

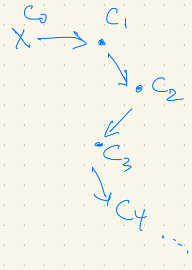
(may require base change $\tilde{V}\to V$)

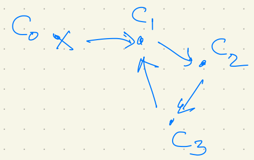
dynamically meaningful sections:

- periodic points



- critical points
- post critical points





PROOF (Trivial or genus 1)

① Choose 3 repelling pts s_1, s_2, s_3 and delete them.

$$\begin{tikzcd} \mathcal{X} \arrow[dr] & = & \mathbb{P}^1-(s_1,s_2,s_3) \arrow[dl] \\ & V & \end{tikzcd}$$

$\simeq V\times\mathbb{P}^1-(0,1,\infty)$

② Use forward orbits of critical points to define sections $c_i(n)=F^n(c_i)$ of $\mathcal{X}\to V$. (uses stability)

③ ∞ many sections $\Longrightarrow$ $c_i(n)$ constant $\Longrightarrow$ F trivial.

④ finitely many $c_i(n)$ $\Longrightarrow$ F critically finite $\Longrightarrow$ (Thurston) F trivial (rigidity) unless F has genus one (Lattès)

Remarks

- F/curve V — good setting for arithmetic dynamics.
- F/Δ^* — study of degenerations. (analyze stable case?)
- Intuition: hyperbolic components $H(f)\subset MRat_d$ are "small".

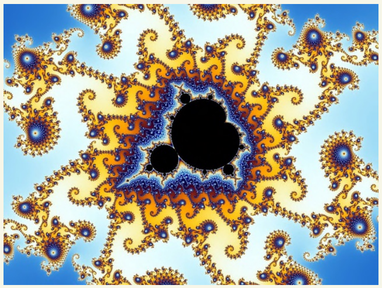
Thm $\Longrightarrow$ no alg curve $V\subset H(f)$ (affine)

Hyperbolicity

A rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is hyperbolic

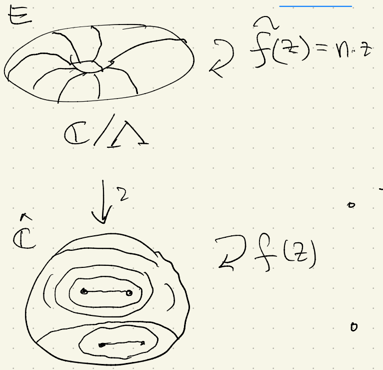
$\iff$ all critical points converge to attracting periodic cycles

$\iff$ $f | J(f)$ is expanding.



MAIN CONJECTURE Hyperbolicity is (open and) dense in Poly_d and Rat_d.

Invariant Line Fields



Lattès construction
"genus one"

- $J(f)$ has positive measure
- $\exists$ measurable invariant line field on $J(f)$.

CONJECTURE A

$J(f)$ carries no invariant line field, unless "genus 1".

$\Rightarrow$ $\boxed{\text{Hyperbolicity is dense.}}$

Progress Work on MLC; $z^{2n}+c$ (M) $c \in \mathbb{R}$

Complements $\exists$ poly_d, $m(J(f))>0$

Stability (Mañé-Sad-Sullivan)

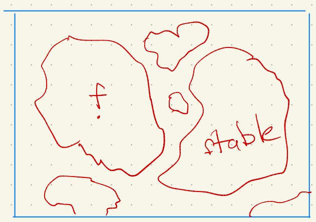
f_0 is stable in f_λ, $\lambda \in \Lambda$ cx mfld

$\iff$ nearby, no bifurcations

$\iff$ $N(f_\lambda)$ = # attr. cycles has local max.

$\iff$ $J(f_\lambda)$ moves by a holomorphic motion nearby.

Thm Stability is open and dense in Poly_d and Rat_d (and Λ).



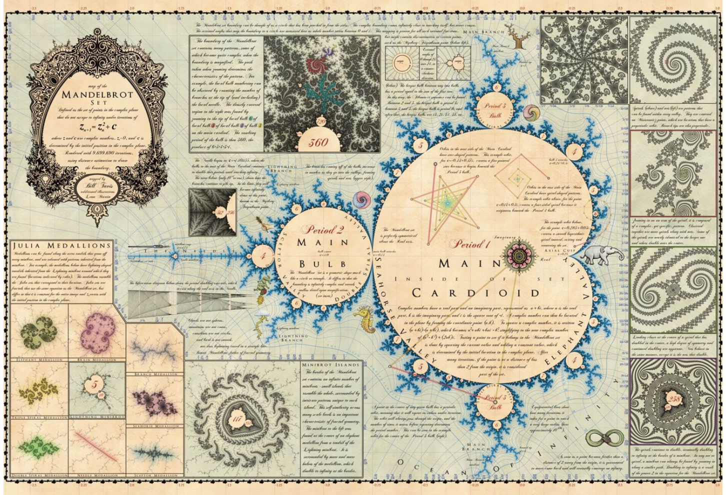
CONJECTURE B

Stability implies hyperbolicity. (in Rat_d or Poly_d).

$\Updownarrow$

Conjecture A

The Mandelbrot Set



Why are the hyperbolic components $H(z^2+c)$ so round?

Program to Study Rational Maps

(cf. M, 1995)

* Assume Hyperbolicity Dense.

* J(f) disconnected ⇝ decompose.

* J(f) connected → H(f) ∋ f_0 critically finite

↓

study topology of f_0, Thurston's thm.

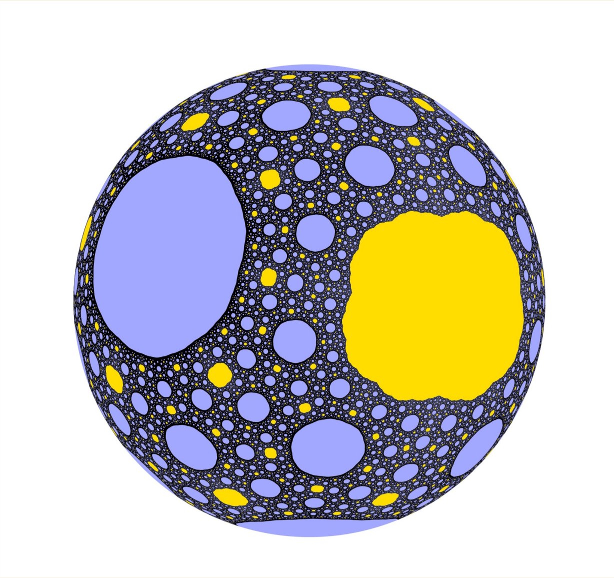
* Further decompose f along cylinders. ←

⇝ crochet, acylindrical

* Study $H(f_0)$, $\partial H(f_0)$, . . .

Acylindrical Rational Map

J(f) is a Sierpinski curve — components of Fatou set do not touch.



Size of Hyperbolic Components

CONJECTURE C If f(z) is acylindrical, then its hyperbolic component has compact closure $\overline{H(f)} \subset \mathrm{MRat}_d$

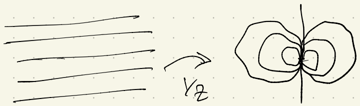
Progress: True in "critically fixed" case. (Dudko, Luo)

Complementary Problem: If f(z) is cylindrical, how can $f_n \to \infty$ in $\overline{H(f)}$?

The Dictionary

Conjs A, B, C known thms for Kleinian groups (Sullivan, Thurston).

* NILF



II. Intermission – Surfaces

Mazur's Problem

Q. Let S be a smooth surface /ℚ. What can one say about $\overline{S(\mathbb{Q})} \subset S(\mathbb{R})$?

Idea: If $F: S \to S$ aut/ℚ, can be used to enlarge $p \in S(\mathbb{Q})$ to $\langle F^n(p) : n \in \mathbb{Z} \rangle$.

Q. Can F be pseudo-Anosov?

Example: K3 surfaces

$F_t : S_t \to S_t$

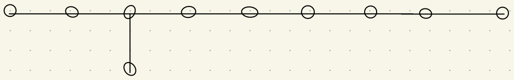
Questions

- * Is $\{t : F_t$ has an elliptic island$\}$ open + dense?
- * Is $\{t : F_t$ has dense orbit in $S_t(\mathbb{R})\}$ of pos. measure?
- * Over $\mathbb{C}$, does $J(F_t) = S_t \ \forall t$?
- * What entropy can arise on proj. K3s?

Progress: $\log \lambda_{Lehmer}$ (M), λ^n (Brandhorst)

Rational Surfaces

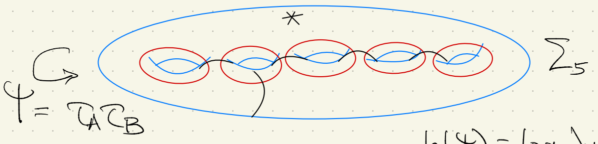
E_{10}



Thm (M) $\exists$ $F : S \to S$ on a rat'l surface with $h(F) = \log \lambda_{Lehmer} = \log(1.17..)$ and no smaller h is possible.

$\overset{\displaystyle h}{\underset{0}{\vee}}$

TOPOLOGY

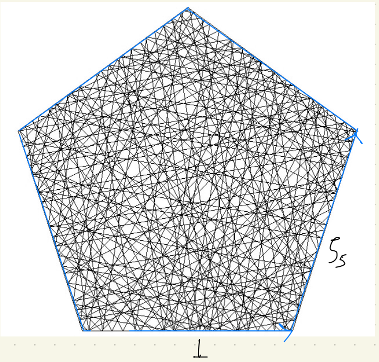


$h(\psi) = \log \lambda_{Lehmer}$

Thm (Diller – Kim)

One can arrange that F, S defined $/\mathbb{R}$ and $F|S(\mathbb{R}) \underset{\text{isotopic}}{\sim} \psi$ on $\Sigma_5 \# \mathbb{RP}^2$.

III. Billiards in polygons.



Regular pentagon P.

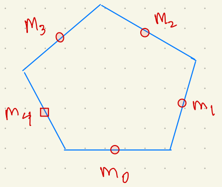
Periodic directions $\Longleftrightarrow$ vectors $z \in \mathbb{Z}[\zeta_5] \subset \mathbb{C}$.

OPEN PROBLEM For regular 7-gon, which $z \in \mathbb{Z}[\zeta_7]$ are periodic directions

Is there an algorithm (of any kind!) to test if a trajectory in direction z is periodic?

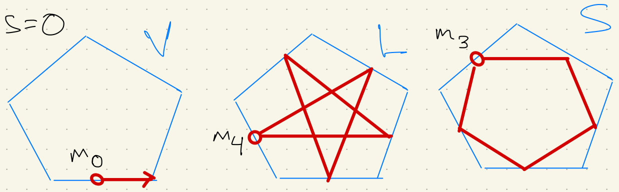
Pentagon Problem

(M, 2018)



- Start trajectory at one of 5 edge midpoints.
- Long, Short, or Vertex-hitting.

Ex. $s=0$



PROBLEM: Given edge midpoint and periodic direction (M_i, s), determine if trajectory has type V, L or S.

Periodic Directions

$L = \mathbb{Q}(\zeta_5)$

$|\,2$

$K = \mathbb{Q}(\sqrt{5}) \supset \mathcal{O} = \mathbb{Z}[\gamma] \quad \gamma = \frac{1+\sqrt{5}}{2}$

$|\,2$

$\mathbb{Q}$. $\qquad L = K \oplus K\zeta_5$.

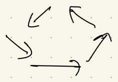
Periodic directions $\longleftrightarrow$ real lines in L

$\longleftrightarrow \quad \mathbb{P}^1 K \simeq \mathbb{P}^1\mathcal{O}$.

(class # 1)

$[a+b\zeta_5] \longleftrightarrow [a:b] \in \mathbb{P}^1(\mathcal{O})$.

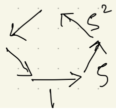
$[\zeta_5^i]$ give 5 pts in $\mathbb{P}^1(\mathcal{O})$



prime 2 inert in K : $\mathcal{O}/2 \simeq \mathbb{F}_4$

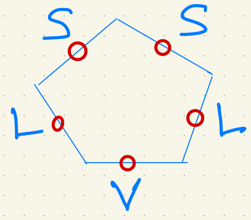
$\mathbb{P}^1(\mathcal{O}/2) \simeq \mathbb{P}^1\mathbb{F}_4 = \mathbb{F}_4 \cup (\infty)$

$= \{\zeta_5^i \bmod 2\}$.



Pentagon Problem — Solution

① Types always appear in same cyclic order:



② Behavior of given direction $s \in \mathbb{P}^1\mathcal{O}$ depends only on $[s] \in \mathbb{P}^1(\mathcal{O}/2)$.

③ Midpoint i has type V $\iff$ $[s] = [\zeta_5^i]$ in $\mathbb{P}^1(\mathcal{O}/2)$.

→ PERIODIC BEHAVIOR

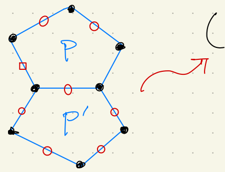
(Cf. Everett, Lin, Mager, 2021) different soln.

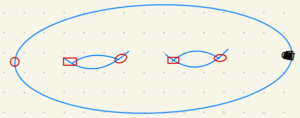
Where are algebraic curves and complex dynamics?

① Riemann surface X, $g=2$

Holo. 1-form $\omega \in \Omega(X)$.

$(X,\omega) = (P \cup P', dz)/\sim$





glue opposite sides

6 Weierstrass points of X

$\longleftrightarrow$ vertices of $P = Z(\omega)$,

$+$ 5 edge midpoints.

② Billiard trajectories $\longleftrightarrow$

flow lines of holomorphic vector field

v on X, $\omega(v) = \exp(i\theta)$.

(holomorphic dynamics)

Teichmüller curves

③ Family of forms $\omega_t = \omega + t\overline{\omega}$,

$t \in$ unit disk Δ.

Unique cx str on X s.t. $\omega_t \in \Omega(X_t)$.

→ Natural holomorphic map

$$\begin{tikzcd} \pi : \Delta \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathcal{M}_g \\ \mathbb{H} \arrow[dr] & \\ & V = \mathbb{H}/\Delta(2,5,\infty) \arrow[uu] \end{tikzcd}$$

$g=2$,

$\pi(t) = X_t$

THM V is a Teichmüller curve, ie an algebraic curve isometrically immersed in $\mathcal{M}_g$.

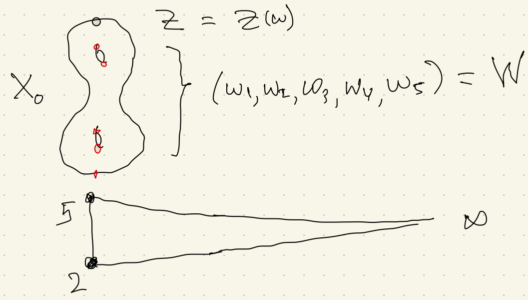
PROBLEM Classify (primitive) Teich. curves.

Unknown: Are there ∞ many in $\mathcal{M}_5$?

Only know construction, $g \geq 5$, Bouw–Möller.

Monodromy (crux)

$$\begin{tikzcd} X \arrow[d] \\ V \end{tikzcd}$$



$\pi_1 V = \Delta(2,5,\infty) \subset SL_2 \mathcal{O} / \pm I$

acts on directions $\simeq \mathbb{P}^1 \mathcal{O}$

acts on Weierstrass pts W

Key is to show:

$$\begin{tikzcd} & \mathbb{P}^1(\mathcal{O}) \arrow[d] \\ \pi_1 V \arrow[ur, squiggly] \arrow[r, squiggly] & W \end{tikzcd}$$

$\mathbb{P}^1(\mathcal{O})$ periodic directions

W 5 edge midpts

Jacobian

$W_i \longmapsto [W_i - Z] \in \mathrm{Jac}(X_0)$

$i = 1, 2, 3, 4, 5.$ ($\leftrightarrow$ pentagon midpts)

$2W_i - 2Z = (f)$ $\quad X_0 \xrightarrow[f]{2} \mathbb{P}^1$

hyperelliptic.

so $W_i \in \mathrm{Jac}(X_0)[2]$

$\simeq H_1(X_0, \mathbb{F}_2) \simeq \mathbb{F}_2^{2g} = \mathbb{F}_2^4$

Also $\mathcal{O} \subset \mathrm{End}\,\mathrm{Jac}(X_0)$

so $W \longrightarrow H_1(X_0, \mathbb{F}_2) \simeq (\mathcal{O}/2)^4 \dashrightarrow \mathbb{P}^1(\mathcal{O}/2).$

Thus

$$\begin{tikzcd} & W \simeq \mathbb{P}^1(\mathcal{O}/2) \\ \pi_1 V \arrow[ur, squiggly, "\text{acts}"'] \arrow[r, squiggly, "\text{acts}"'] & \mathbb{P}^1(\mathcal{O}) \arrow[u, "\text{quotient}"'] \end{tikzcd}$$

periodic directions

Dihedral Group

$\Delta(2,5,\infty) \subset SL_2(\mathcal{O})$

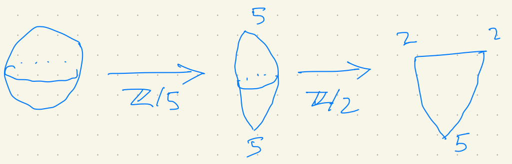
$\rightarrow SL_2(\mathcal{O}/2),$

acts on $\mathbb{P}^1(\mathcal{O}/2) \simeq W.$

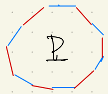
mod 2, parabolics become of order 2:

$\therefore$ Image in $SL_2(\mathcal{O}/2)$ is

$\Delta(2,5,2) \simeq D_{10}$



Higher Genus — Arithmetic Chaos



Regular 12-gon

$\rightarrow (X, \omega) = (P, dz)/\sim$

$g(X) = 3.$ $\quad \# \mathrm{WP}(X) = 8$

$\cup$

$\#(\text{edge mpts } M) = 6.$

$V = \mathbb{H} / \Delta(6, \infty, \infty) \subset M_3$

Teich curve.

$\Delta(6, \infty, \infty) \subset SL_2 \mathcal{O}$

$\mathcal{O} = \mathbb{Z}[\sqrt{3}] \subset K \overset{2}{\subset} L = \mathbb{Q}(\zeta_{12})$

Periodic directions $\longleftrightarrow \mathbb{P}^1(\mathcal{O})$

$\longleftrightarrow$ vectors in $\mathbb{Z}[\zeta_{12}].$

Dodecagon Problem

Given $s \in \mathbb{P}^1\mathcal{O}$, predict midpoint m_i such that trajectory from m_i at slope s hits a vertex.

PROBLEM $\underset{\text{midpts}}{M} \longrightarrow \mathbb{P}^1(\mathcal{O}/2)$ is 2-1.

Thm M does not map 1-1 into $\mathbb{P}^1(\mathcal{O}/n)$ for any n.

There is always at least a $\mathbb{Z}/2$ ambiguity, $\varepsilon(s)$.

Interpreation: ℓ-adic reps

Monodromy $\pi_1(V, X_0) \longrightarrow \text{Aut } H_1(X_0, \mathbb{Z}/(\ell^n$

ℓ prime

$\Rightarrow$ limit as $n \to \infty$: ℓ-adic rep

$\pi_1(V) = \Delta(6, \infty, \infty) \longrightarrow Sp_{2g}(\mathbb{Z}_\ell)$.

Cf. Galois Reps.

Theme: "Monodromy is as large as possible." open or finite index

Serre, Ribet, ...

Our case $\Delta \to SL_2(\mathcal{O}) \to SL_2(\mathcal{O}_P)$

for each prime P of $\mathcal{O}$.

Thm $\Delta \to \prod_P SL_2(\mathcal{O}_P) = SL_2(\hat{\mathcal{O}})$

is "as large as possible".

Cor M does not inject into $\mathbb{P}^1(\hat{\mathcal{O}})$.

$\Rightarrow$ Dodecagon problem has no solution.

Project: Describe $\overline{\Delta} \subset SL_2\hat{\mathcal{O}}$ for all triangle groups $\Delta(p, q, r)$.

Adelic Theory of Triangle Groups

$\Delta = \Delta(2, n, \infty)$ or $\Delta(n, \infty, \infty)$ ... $\subset SL_2\mathcal{O}$

$\overline{\Delta}$ = closure in $SL_2(\hat{\mathcal{O}})$.

$= \prod \Delta_p$, $p = 2, 3, 5, \ldots$

Ex. $\Delta = \Delta(2, 5, \infty)$

THM $[SL_2(\hat{\mathcal{O}}) : \overline{\Delta}] = 72$

$= \begin{pmatrix}\text{index } 12 \\ \text{at } p=2\end{pmatrix} \times \begin{pmatrix}\text{index } 6 \\ \text{at } p=3\end{pmatrix}$

$\underset{60}{SL_2\mathbb{F}_4} \supset D_{10}$ $\qquad$ $\underset{720}{SL_2\mathbb{F}_9} \supset \underset{120}{\tilde{A}_5}$

Conclusion: Arithmetic Chaos

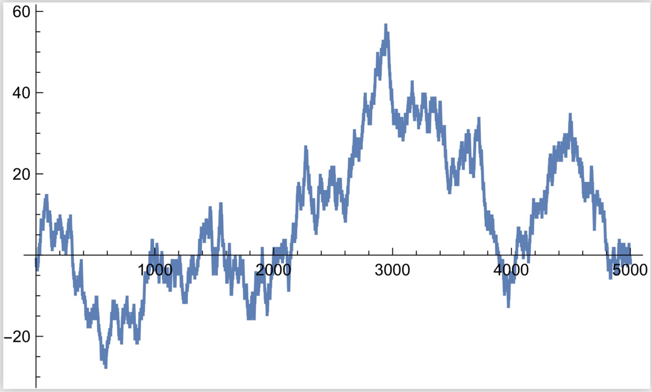
We have (missing) invariant (homomorphism!)

$$\begin{tikzcd}
\varepsilon : \Delta(6,\infty,\infty) \arrow[rr] \arrow[dr] & & \mathbb{Z}/2 = (\pm 1) \\
& \mathbb{P}^1\mathcal{O} \arrow[ur] &
\end{tikzcd}$$

Does not factor through $\mathbb{P}^1(\mathcal{O}/n)$, any n. $\Rightarrow$ $\varepsilon(k)$ $k = 1, 2, \ldots$ not periodic. How does it behave?

Conjecture: The sequence $\varepsilon(k)$ behaves like a random walk: mean zero, central limit thm, etc.

Experimental Data



Plot of $W_n = \sum_{k=1}^{n} \varepsilon(k) = \pm 1$.

Regular Polygons

- Billiards
- Teichmüller curves
- ℓ-adic represenations
- deterministic chaos of arithmetic origin