Dynamics on Riemann surfaces and the geometry of moduli space

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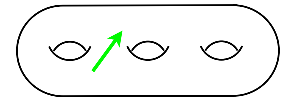
$SL_2(\mathbb{R}) \circlearrowleft \Omega\mathcal{M}_g$ $\mathbb{H}/\Gamma \to \mathcal{M}_g \to B/\Gamma$

Avila, Hubert, Lanneau, Kontsevich, Masur, Yoccoz, Zorich, ...

Hyperbolic surfaces

X genus g
Riemann surface

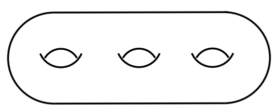


Hyperbolic metric $\rho^2(z)\,|dz|^2$

- Geodesic flow on T_1X
- Ergodic, mixing, entropy > 0
- # Loops $L(C) < L \sim e^L/L$
- $\mathrm{Aut}(X,\rho)$ finite
- Charts into $\mathbb{H}$

Flat metrics

(X,ω)



$\Omega(X)$ = {holomorphic forms $\omega(z)\,dz$} $\simeq \mathbb{C}^g$

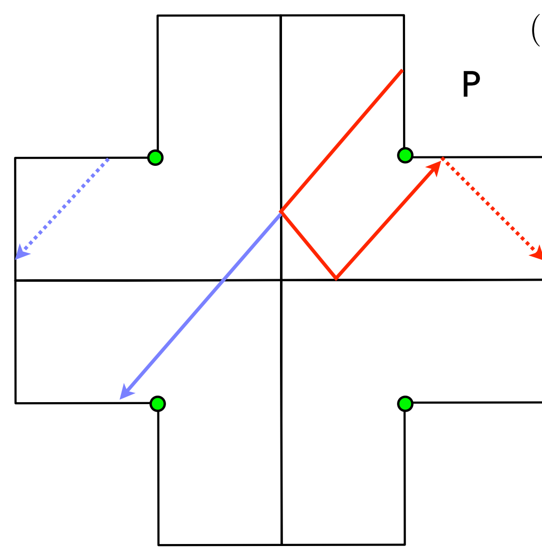
$\omega \in \Omega(X) \Rightarrow$ flat metric $|\omega|^2\,|dz|^2$

$\omega(p)=0$: negatively curved singularities

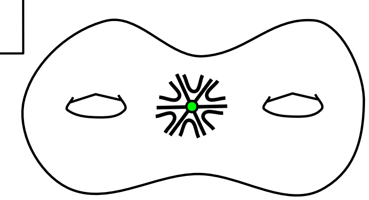
- Geodesics with fixed slope foliate X
- Not mixing, entropy = 0
- # Smooth Loops $L(C) < L \asymp L^2$
- Charts into $\mathbb{C} \simeq \mathbb{R}^2$ up to translation

Example: Billiards

$$(X,\omega) = \left(\bigcup \rho P, dz\right) / \sim$$



X has genus 2
ω has just one zero!



Real Symmetries

Aff(X,ω) = {f : X→X, *real linear maps*}

SL(X,ω) = {Df } = Γ ⊂ $SL_2(\mathbb{R})$ *discrete*

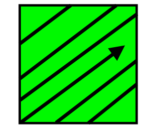
Theorem (Veech): *If* SL(X,ω) *is a lattice, then the geodesic flow has optimal dynamics.*

Proof: *renormalization.*

Optimal Billiards

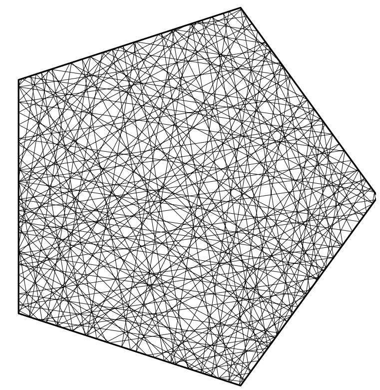
Example: if X = ℂ/Λ, ω=dz, then

SL(X,ω) ≃ $SL_2(\mathbb{Z})$



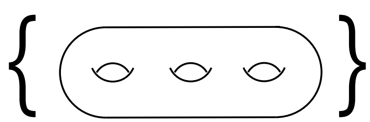
Theorem (Veech, 1989): *For (X,ω) = ($y^2 = x^n-1$, dx/y), SL(X,ω) is a lattice.*

Corollary
A billiard path in a regular polygon is periodic or uniformly distributed.



Moduli space perspective

$\mathcal{M}_g$ = the moduli space of Riemann surfaces X of genus g



-- a complex variety, dimension 3g-3

Teichmüller metric: every holomorphic map

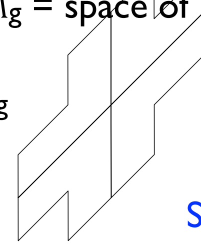
$$f : \mathbb{H}^2 \to \mathcal{M}_g$$

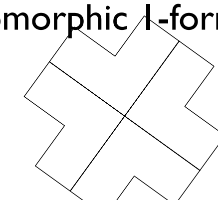
is distance-decreasing.

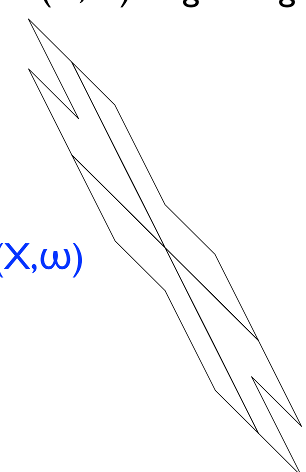
= *Kobayashi metric*

Dynamics over moduli space

$\Omega\mathcal{M}_g$ = space of holomorphic 1-forms (X,ω) of genus g
↓
$\mathcal{M}_g$

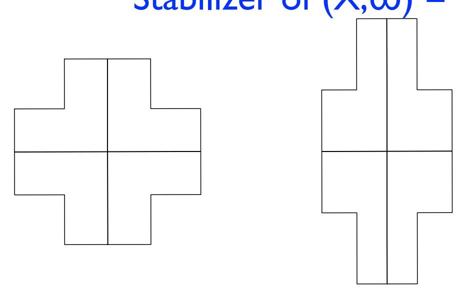






$SL_2(\mathbb{R})$ acts on $\Omega\mathcal{M}_g$

Stabilizer of (X,ω) = SL(X,ω)



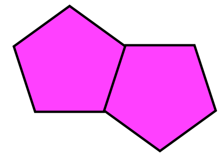
Teichmüller curves

$SL_2(\mathbb{R})$ orbit of (X,ω) in $\Omega\mathcal{M}_g$ projects to a *complex geodesic* in $\mathcal{M}_g$:

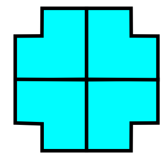
$$\begin{array}{ccc} \mathbb{H} & \longrightarrow & \mathcal{M}_g \\ & \searrow \quad \nearrow f & \\ & V = \mathbb{H} \,/\, SL(X,\omega) & \end{array}$$

$SL(X,\omega)$ lattice $\Leftrightarrow$ $f : V \to \mathcal{M}_g$ is an algebraic, isometrically immersed *Teichmüller curve*.

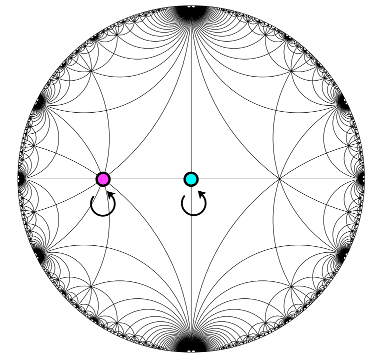
Explicit package: Pentagon example



$(X,\omega) = (y^2 = x^5 - 1, dx/y)$



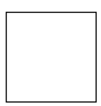
$g \cdot (X,\omega)$



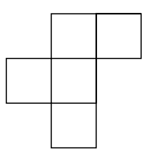
$$\begin{array}{c} \mathcal{M}_g \\ \uparrow \\ V = \mathbb{H}/SL(X,\omega) \subset SL_2(\sqrt{5}) \\ = \langle a, b \rangle \end{array}$$

⇒ Direct proof that $SL(X,\omega)$ is a lattice

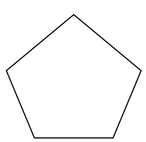
20th century lattice billiards



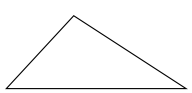
Square SL_2(Z)



Tiled by squares ~ SL_2(Z)



Regular polygons ~ (2,n,∞) triangle group



Various triangles triangle groups

Genus 2

⤳ Regular 5- 8- and 10-gon

Problem

Are there infinitely many primitive Teichmüller curves V in the moduli space $\mathcal{M}_2$?

Jacobians with real multiplication

Theorem

(X,ω) generates a Teichmüller curve $V \Rightarrow$

$Jac(X)$ admits real multiplication by $\mathcal{O}_D \subset \mathbb{Q}(\sqrt{D})$.

Corollary

V lies on a Hilbert modular surface

$$V \subset H_D \subset \mathcal{M}_2$$
$$H_D = \mathbb{H} \times \mathbb{H} / SL_2(\mathcal{O}_D)$$

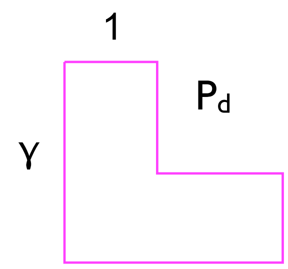
Idea of Proof: $f+f^{-1}$ acts on $Jac(X) \Rightarrow$

trace ring of $SL(X,\omega)$ acts

The Weierstrass curves

$W_D = \{X \text{ in } \mathcal{M}_2 :\ Jac(X)$ admits real multiplication by $\mathcal{O}_D$ with an eigenform ω with a double zero.$\}$

Theorem. *W_D is a finite union of Teichmüller curves.*

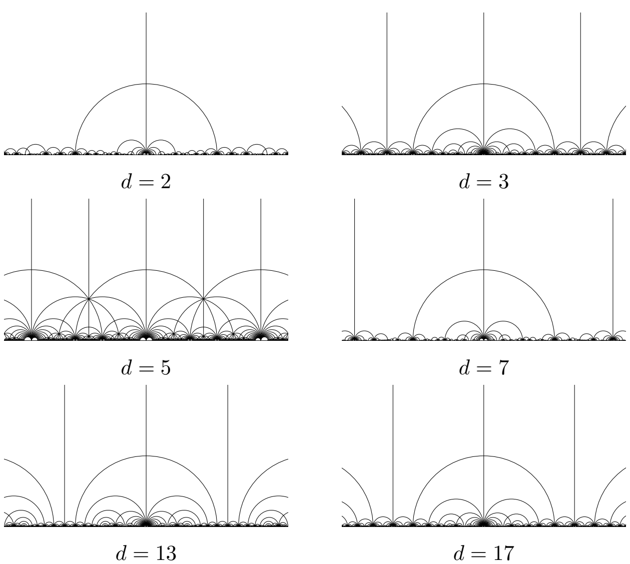


$\gamma = (1 + \sqrt{d})/2$

Corollaries

- *P_d has optimal billiards for all integers $d>0$.*
- *There are infinitely many primitive V in genus 2.*

Examples of Weierstrass curves



$W_D \supset \mathbb{H}/\Gamma$

Γ not arithmetic

no known direct description of Γ

← *algorithm only works if $g(W_D) = 0$*

Components of W_D

Theorem. *W_D is irreducible unless $D=1 \bmod 8$ and $D>9$, in which case it has two components.*

(spin)

Proof

a) combinatorial enumeration of cusps of W_D

b) elementary move relating cusps in same component $\Rightarrow$ graph S_D

c) proof that S_D is connected
(analytic number theory + computer for $D < 2000$)

Euler characteristic of W_D

Theorem (Bainbridge, 2006)

$$\chi(W_D) = -\frac{9}{2}\chi(\mathrm{SL}_2(\mathcal{O}_D))$$

= *coefficients of a modular form*

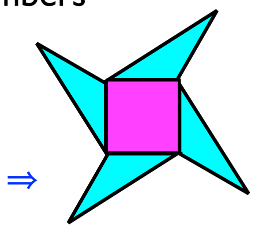
Compare: $\chi(\mathcal{M}_{g,1}) = \zeta(1-2g)$ (Harer-Zagier)

Proof: Uses cusp form on Hilbert modular surface with $(\alpha) = W_D - P_D$, where P_D is a Shimura curve

Elliptic points on W_D

Theorem (Mukamel, 2011)

The number of orbifold points (of order 2) on W_D is given by a sum of class numbers for $\mathbb{Q}(\sqrt{-D})$.



Proof: (X,ω) correponds to an orbifold point $\Rightarrow$

X covers a CM elliptic curve E $\Rightarrow$

(X,ω), $p: X \to E$ and $Jac(X)$ can be described explicitly.

Genus of W_D

D	$g(W_D)$	$e_2(W_D)$	$C(W_D)$	$\chi(W_D)$
5	0	1	1	$-\frac{3}{10}$
8	0	0	2	$-\frac{3}{4}$
9	0	1	2	$-\frac{1}{2}$
12	0	1	3	$-\frac{3}{2}$
13	0	1	3	$-\frac{3}{2}$
16	0	1	3	$-\frac{3}{2}$
17	$\{0,0\}$	$\{1,1\}$	$\{3,3\}$	$\{-\frac{3}{2},-\frac{3}{2}\}$
20	0	0	5	-3
21	0	2	4	-3
24	0	1	6	$-\frac{9}{2}$
25	$\{0,0\}$	$\{0,1\}$	$\{5,3\}$	$\{-3,-\frac{3}{2}\}$
28	0	2	7	-6
29	0	3	5	$-\frac{9}{2}$
32	0	2	7	-6
33	$\{0,0\}$	$\{1,1\}$	$\{6,6\}$	$\{-\frac{9}{2},-\frac{9}{2}\}$
36	0	0	8	-6
37	0	1	9	$-\frac{15}{2}$
40	0	1	12	$-\frac{21}{2}$
41	$\{0,0\}$	$\{2,2\}$	$\{7,7\}$	$\{-6,-6\}$
44	1	3	9	$-\frac{21}{2}$
45	1	2	8	-9
48	1	2	11	-12
49	$\{0,0\}$	$\{2,0\}$	$\{10,8\}$	$\{-9,-6\}$

D	$g(W_D)$	$e_2(W_D)$	$C(W_D)$	$\chi(W_D)$
52	1	0	15	-15
53	2	3	7	$-\frac{21}{2}$
56	3	2	10	-15
57	$\{1,1\}$	$\{1,1\}$	$\{10,10\}$	$\{-\frac{21}{2},-\frac{21}{2}\}$
60	3	4	12	-18
61	2	3	13	$-\frac{33}{2}$
64	1	2	17	-18
65	$\{1,1\}$	$\{2,2\}$	$\{11,11\}$	$\{-12,-12\}$
68	3	0	14	-18
69	4	4	10	-18
72	4	1	16	$-\frac{45}{2}$
73	$\{1,1\}$	$\{1,1\}$	$\{16,16\}$	$\{-\frac{33}{2},-\frac{33}{2}\}$
76	4	3	21	$-\frac{57}{2}$
77	5	4	8	-18
80	4	4	16	-24
81	$\{2,0\}$	$\{0,3\}$	$\{16,14\}$	$\{-18,-\frac{27}{2}\}$
84	7	0	18	-30
85	6	2	16	-27
88	7	1	22	$-\frac{69}{2}$
89	$\{3,3\}$	$\{3,3\}$	$\{14,14\}$	$\{-\frac{39}{2},-\frac{39}{2}\}$
92	8	6	13	-30
93	8	2	12	-27
96	8	4	20	-36

Corollary

W_D has genus 0 only for $D<50$

(table by Mukamel)

Explicit points on W_D

	$X \in \mathcal{M}_2$
D=5	$y^2 = x^5-1$
D=8	$y^2 = x^8-1$
D=13	$y^2 = (x^2-1)(x^4-ax^2+1)$
	$a = 2594 + 720\sqrt{13}$
	....
D=108	$96001 + 48003a + 3a^2 + a^3 = 0$

Mukamel

Genus 3 or more

Algebraically primitive case:

Trace field K of SL(X,ω) has degree g=g(X) over $\mathbb{Q}$.)

(avoids echos of lower genera)

Theorem (Möller)

- Jac(X) admits real multiplication by K,
- P-Q is torsion in Jac(X) for any two zeros of ω.

Methods: Variation of Hodge structure; rigidity theorems of Deligne and Schmid; Neron models; arithmetic geometry

Finitness conjecture:

There are only finitely many algebraically primitive Teichmüller curves in $\mathcal{M}_g$, g=3 or more.

Theorem (Möller, Bainbridge-Möller)

Holds for hyperelliptic stratum (g-1,g-1)
Holds for g=3 stratum (3,1)

Conjectures on dynamics on $\Omega\mathcal{M}_g$

I. Every $SL_2(\mathbb{R})$ orbit-closure and every $SL_2(\mathbb{R})$ ergodic measure on $\Omega\mathcal{M}_g$ is algebraic.

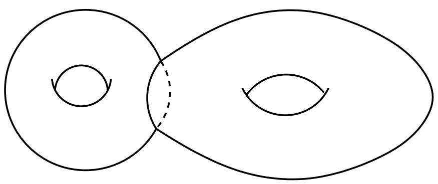
II. The closure of any complex geodesic $f(\mathbb{H})$ is an algebraic subvariety of $\mathcal{M}_g$.

Celebrated theorem of Ratner (1995) ⇒ true for $SL_2(\mathbb{R})$ acting on G/Γ

Genus two

Theorem *These conjectures hold for genus g=2.*

Proof: 1) Any 1-form (X,ω) of genus 2 is a connect sum of forms of genus 1



2) Ratner's theorem holds for diagonal unipotent actions on $SL_2(\mathbb{R})\times SL_2(\mathbb{R}) / SL_2(\mathbb{Z})\times SL_2(\mathbb{Z})$.

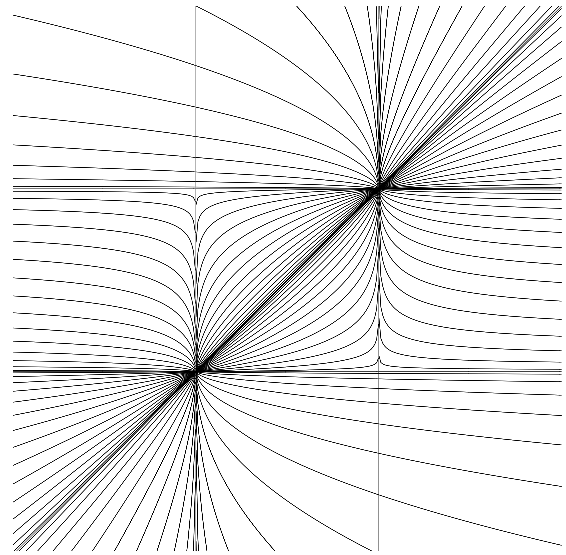
Complex geodesics in genus two

Theorem

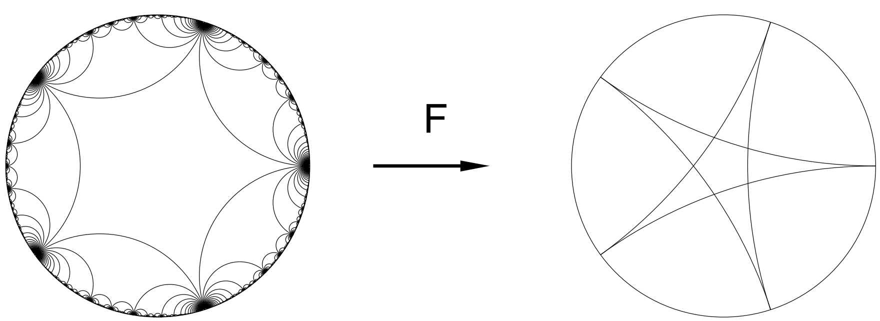
Let $f : \mathbb{H} \to \mathcal{M}_2$ be a complex geodesic generated by $(X,\omega) \in \Omega\mathcal{M}_g$. Then $\overline{f(\mathbb{H})}$ is either:

	dim
• A Teichmüller curve (such as W_D),	1
• A Hilbert modular surface H_D, or	2
• The whole space $\mathcal{M}_2$.	3

Hilbert modular surface in M_2 is foliated by complex geodesics $\simeq \mathbb{H}$



How W_5 sits on H_5



W_5

The universal cover of W_5 = the graph of F in the universal cover $\mathbb{H}$x$\mathbb{H}$ of H_5

(closed leaf of the foliation)

Dynamics on $\Omega\mathcal{M}_g$, g>2

Conjecture

I. Every $SL_2(\mathbb{R})$ orbit-closure and every $SL_2(\mathbb{R})$ ergodic measure on $\Omega\mathcal{M}_g$ is algebraic.

Measure case : recent progress by Eskin and Mirzakhani : Clay Meeting 16 May 2011

Orbit closures still open(?)

Complex geometry of moduli space

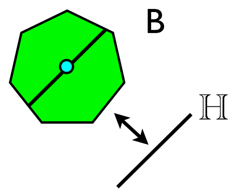
$B \subset \mathbb{C}^n$ bounded domain

Kobayashi metric : Every holomorphic map $\mathbb{H} \to (B, g_K)$ is contracting

Carathéodory metric: Every holomorphic map $(B, g_C) \to \mathbb{H}$ is contracting

Theorem: $B = G/K$ a symmetric domain $\Rightarrow g_K = g_C$

Proof: B has a convex model



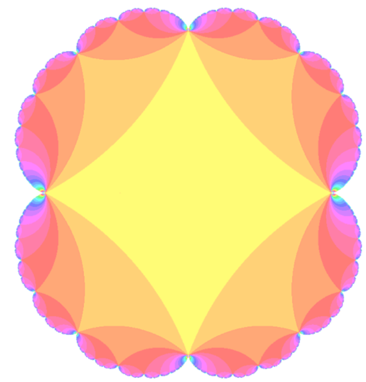
Bers embedding

$\mathcal{M}_g = \mathcal{T}_g / \mathrm{Mod}_g$

$\mathcal{T}_g \hookrightarrow \mathbb{C}^{3g-3}$ as a bounded domain

Open Problem: Does $g_K = g_C$ on Teichmüller space?

Image of $\mathcal{T}_g$



Lots of cusps not convex or starlike

(Dumas)

Related Questions

I. Does $\mathcal{M}_g$ embed isometrically into an infinite product of locally symmetric spaces?

(for the Kobayshi metric)

II. Is the super period map $\mathcal{M}_g \to \prod \mathcal{A}_h$ an isometric embedding?

$X \to$ (Jac(Y) :Y is a finite cover of X)

Theorem (Kra)
The super period map is an isometry on all complex geodesics $\mathbb{H} \to \mathcal{M}_g$ generated by 1-forms (X, ω).

[Hence on all T. curves $V \subset \mathcal{M}_g$

Kazhdan's Theorem

$$\begin{array}{l} \mathbb{H} \\ \downarrow \\ Y \to \mathrm{Jac}(Y) = \mathbb{C}^h/\Lambda \\ \downarrow \\ X \end{array}$$

The hyperbolic metric on X is the limit of the metrics inherited from the Jacobians of finite covers of X.

However...

Theorem.

The super period map

$$\mathcal{M}_g \to \prod_{\mathcal{C}} \mathcal{A}_h$$

is not an isometry in the directions coming from quadratic differentials with odd order zeros.

Corollary.

The entropy of most mapping classes

$f : \Sigma_g \to \Sigma_g$

cannot be detected homologically, even after passing to finite covers.

$\langle$Spectral Gap$\rangle$

Entropy on topological surfaces

$$\mathrm{Mod}_g = \{f : \Sigma_g \to \Sigma_g\}/\text{isotopy} = \pi_1(\mathcal{M}_g)$$

$$h(f) = \min \{\text{entropy of } g : g \text{ isotopic to } f\}$$

= length of loop on moduli space represented by [f].

$\geq$ log spectral radius of f^* on $H^1(\Sigma_g)$

Corollary.

The entropy of most mapping classes

$f : \Sigma_g \to \Sigma_g$

cannot be detected homologically, even after passing to finite covers.

means

$$\begin{array}{ccc} \Sigma_h & \xrightarrow{F} & \Sigma_h \\ \downarrow & & \downarrow \\ \Sigma_g & \xrightarrow[f]{} & \Sigma_g \end{array}$$

$h(f) >$ sup log spectral radius of F^* on $H^1(\Sigma_h)$, over all finite covers.

topological proof? *Koberda*