Billiards, heights and modular symbols

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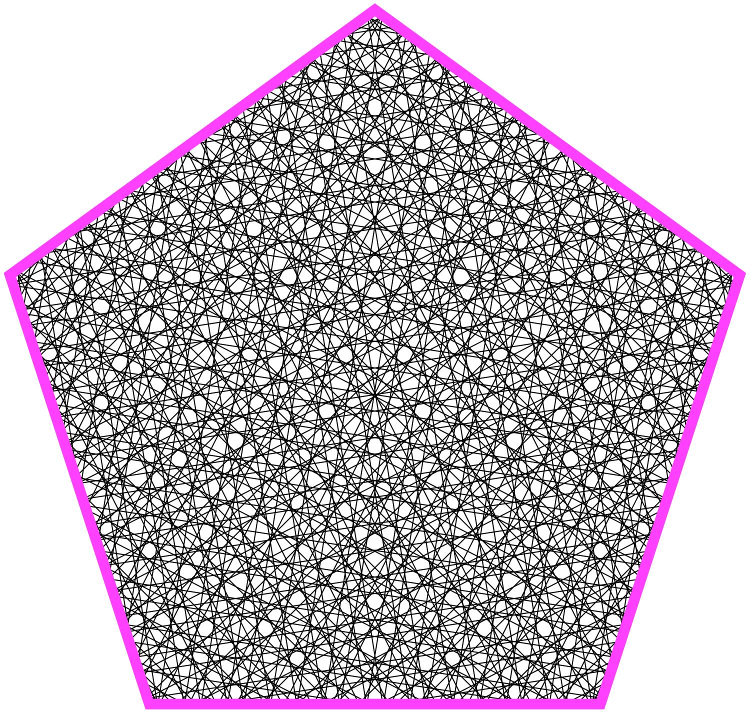
Weil, Manin, Birch, Leutbecher, Veech, Masur, Forni, Möller, Leininger, Hubert, Lanneau, Davis, Lelievre, ….

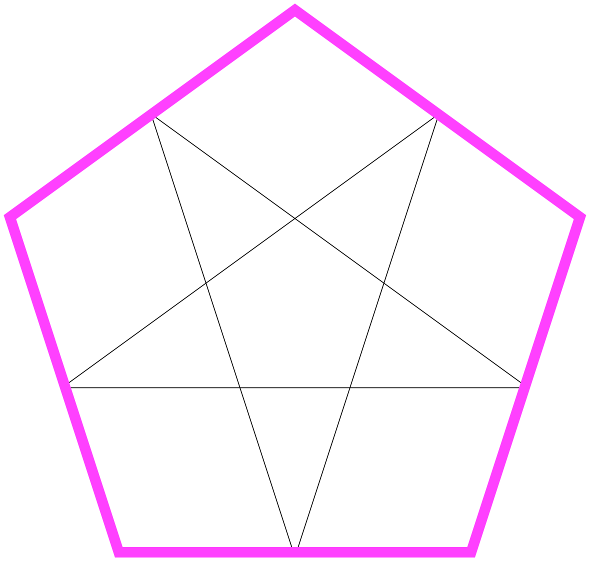
Billiards in a regular pentagon



A dense set of slopes are periodic.

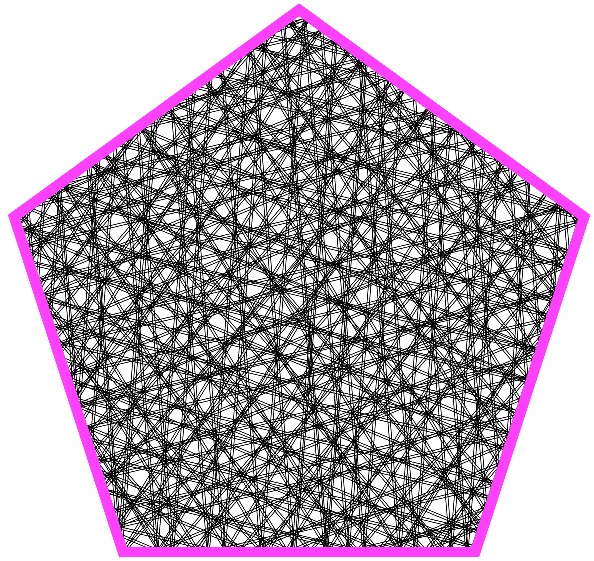
How do the periodic trajectories behave?

Slopes and lengths



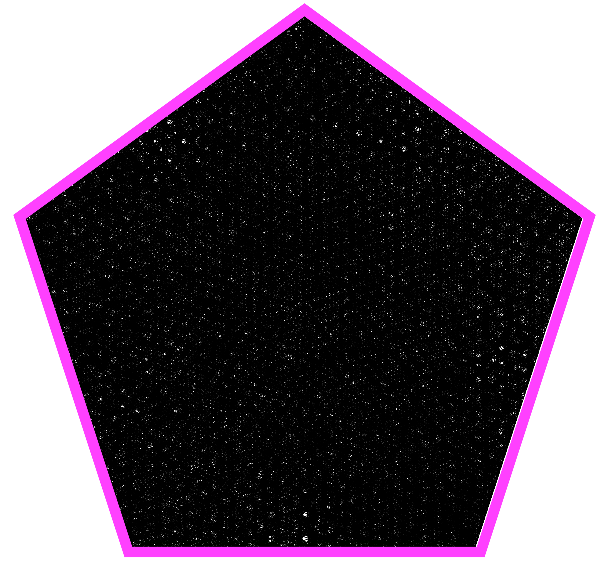
s

L(s) = 5



4s

L(s) = 469



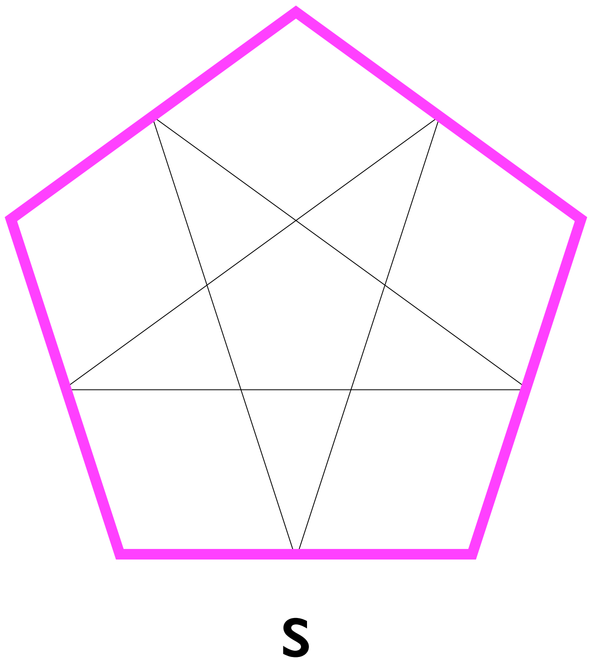
20s

L(s) = 2338

6765s

L(s) = 10^{25}

Slopes, lengths and heights



Theorem 1

The periodic slopes coincide with $\mathbb{Q}(\sqrt{5})s$*, and* $\log L(xs) = O(h(x)^2)$.

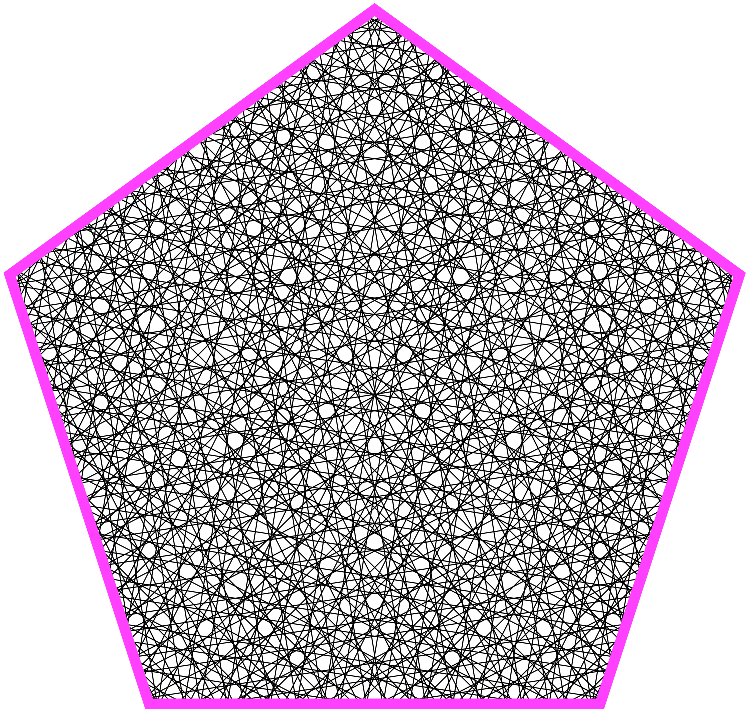
Example

$$L(10^n s) = O(10^{Cn^2})$$

exponent 2 is sharp

Method: *descent on a Hilbert modular surface*

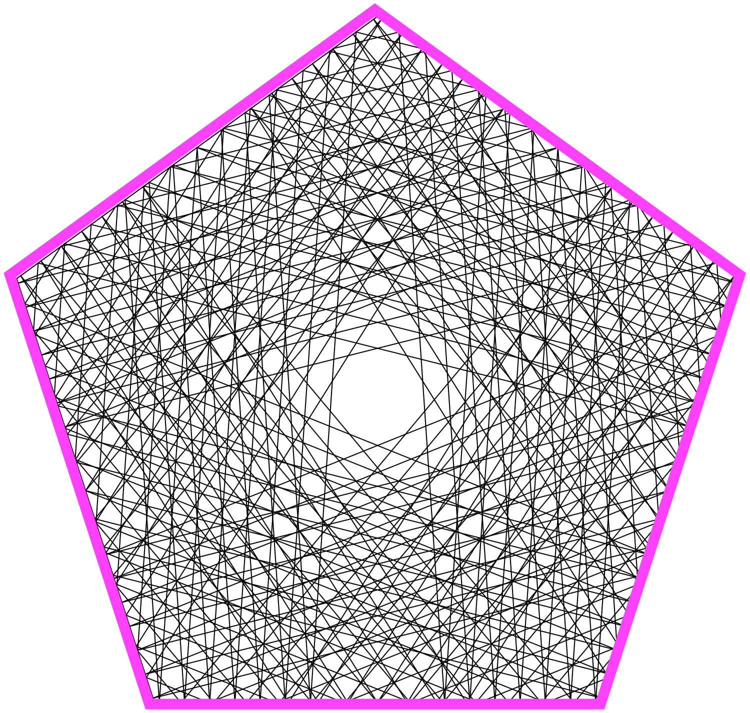
Billiards in a regular pentagon



Every trajectory is
periodic or uniformly distributed.

How are the periodic trajectories distributed?

Billiards in a regular pentagon



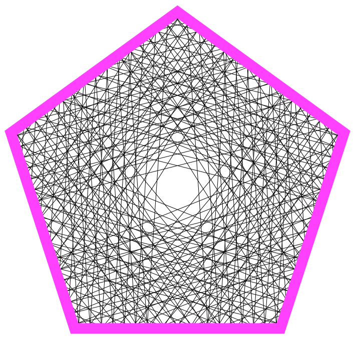
Every trajectory is
periodic or uniformly distributed.

How are the periodic trajectories distributed?

Davis-Lelievre: *Not always uniformly!*

Limit Measures

describe scarring



Theorem II

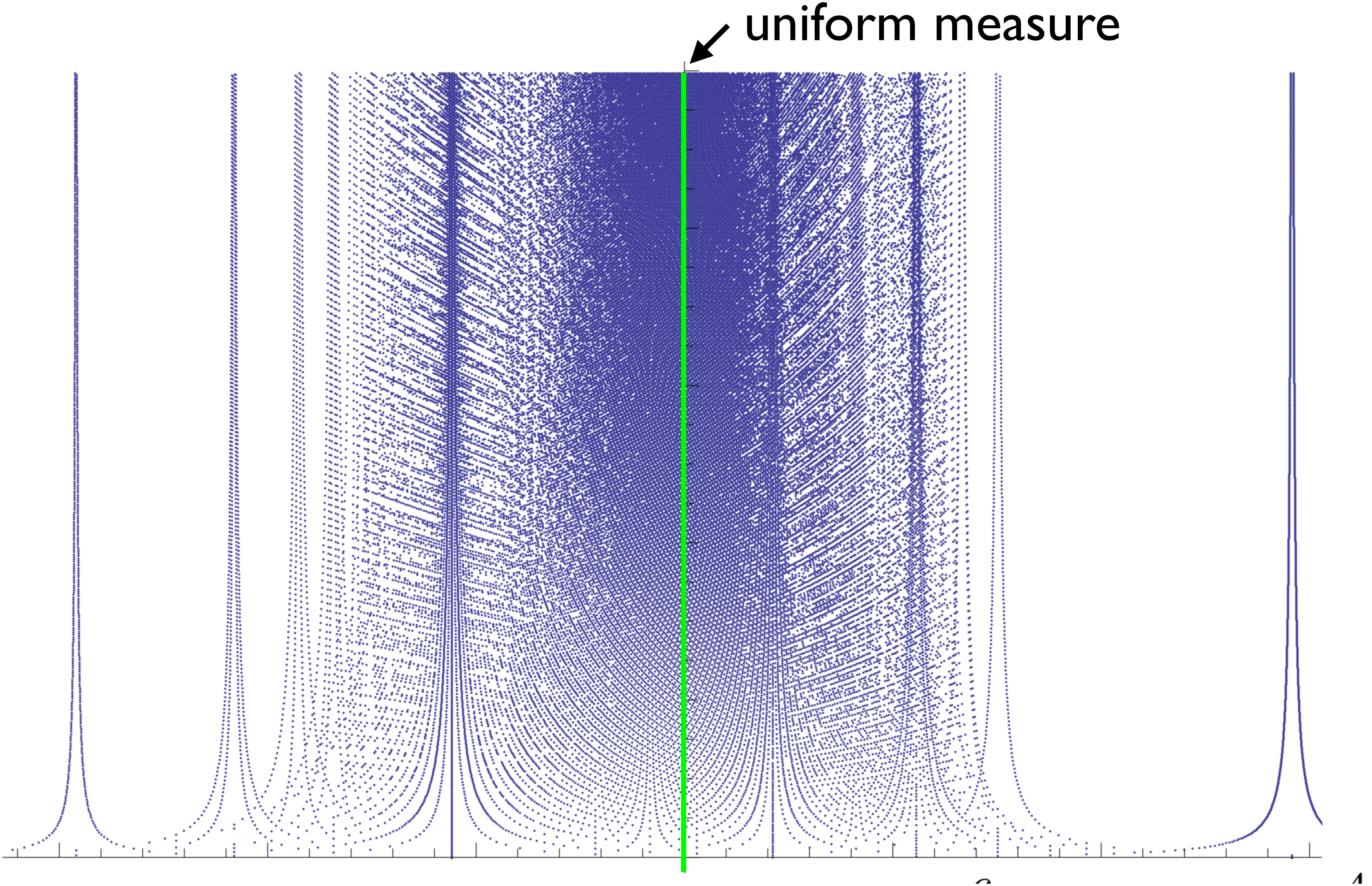
For each periodic slope s, the limit measures M_s form a countable set, homeomorphic to $\omega^\omega + 1$.

Complement

We have uniform distribution iff the lengths of the golden continued fractions of the slopes tend to infinity.

Method: *modular symbols for Teichmüller curves*

Limit Measures M_0



Modular symbols

$V = \mathbb{H}/\Gamma$ hyperbolic surface

modular symbol of degree d: formal product

$$\sigma = \gamma_1 * \gamma_2 * \;\ldots\; * \gamma_d$$

$a_0, a_1, \ldots, a_d$ = *cusps of V*

γ_i *geodesic from* a_{i-1} *to* a_i

$$a_0 \xrightarrow{\gamma_1} a_1 \xrightarrow{\gamma_2} a_2 \quad \cdots\cdots \xrightarrow{\gamma_i} \quad a_d$$

Modular symbols

$$\mathbb{S}(V) \quad = \quad \bigcup_{\textit{degree } d} \mathbb{S}^d(V)$$

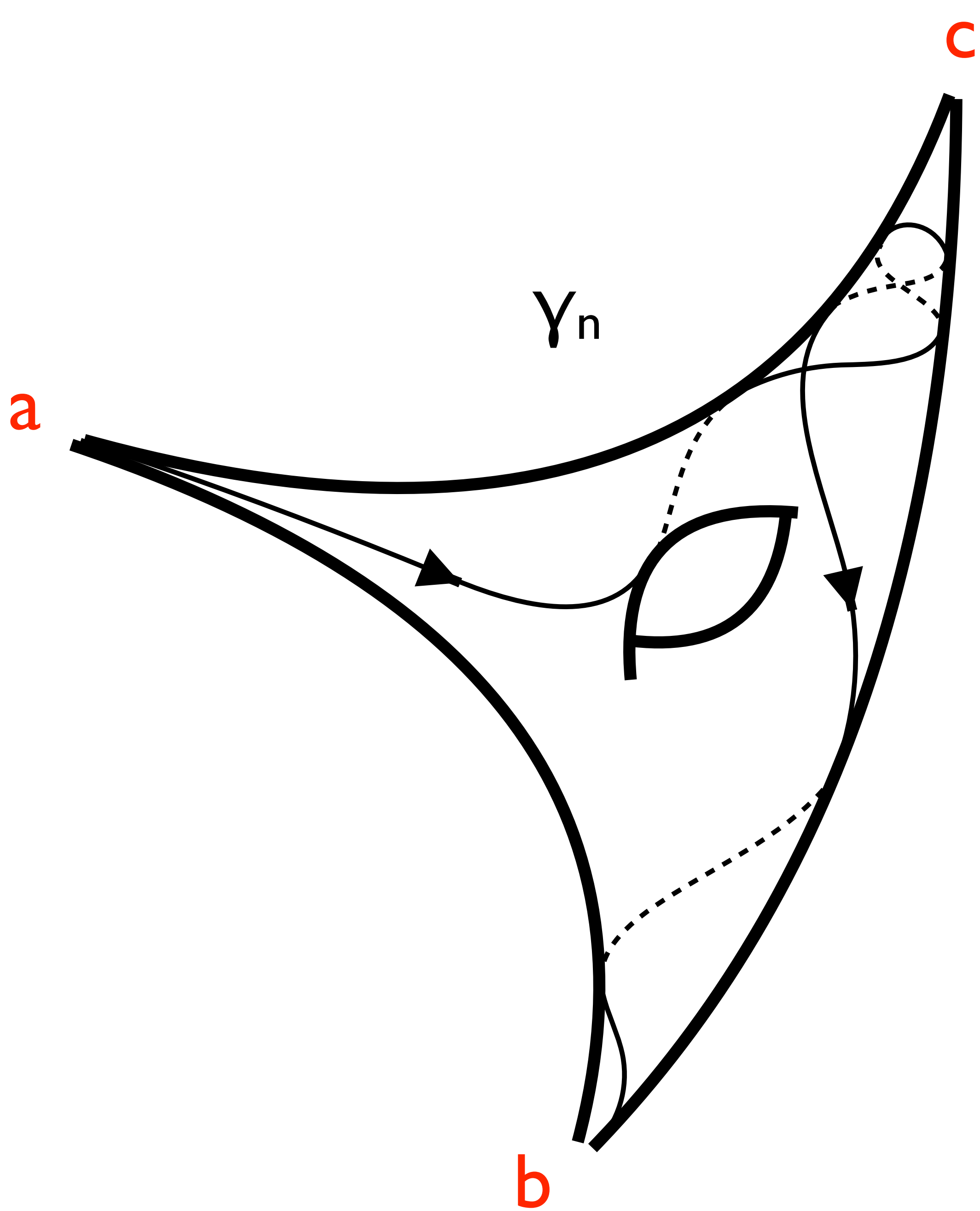
= morphisms in a graded category
whose objects are the cusps of V

geometric topology

$$\overline{\mathbb{S}^d(V)} \quad = \quad \bigcup_{e \geq d} \mathbb{S}^e(V)$$

$$\boxed{\mathbb{S}(V) \simeq \omega^\omega}$$

Modular symbols: topology

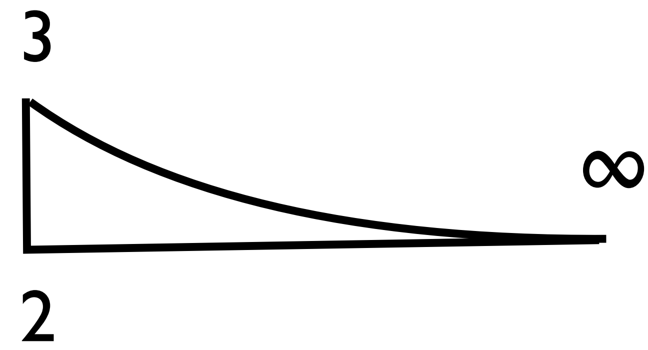


$$\gamma_n \longrightarrow \delta_1 * \delta_2$$

$$a \xrightarrow{\delta_1} c \xrightarrow{\delta_2} b$$

Modular symbols for $V = \mathbb{H}/SL_2(\mathbb{Z})$

$\infty \xrightarrow{\gamma}$ *p/q in [0,1]*



$\mathbb{S}^1(V) = \{ [a_1, \ldots, a_n] \} = \{ 1/a_1 + 1/a_2 + \ldots + 1/a_n \}$

$\mathbb{S}(V) = \{ [a_1, \ldots, a_n] : \text{some } a_i = \infty \}$

$[a_1, \ldots, a_n] * [b_1, \ldots, b_m] = [a_1, \ldots, a_n, \infty, b_1, \ldots, b_m]$

$\{ [a_1, \ldots, a_n] : n \leq N \}$ is *compact*

Aside: Classical Modular symbols

$\mathbb{Q} \cup \infty$ = cusps of $\Gamma(N)$ in $SL_2(\mathbb{Z})$

$X(N)$ = completion of $\mathbb{H} / \Gamma(N)$

$$\{p,q\} : \mathbb{Q}\times\mathbb{Q} \longrightarrow \Omega(X)^* \simeq H_1(X(N), \mathbb{R})$$

abelian

Theorem (Manin-Drinfeld)

The difference of any 2 cusps of X(N) is torsion in Jac(X(N)).

Teichmüller curves

(X,ω) = holomorphic 1-form of genus g

$V = \mathbb{H} / SL(X,\omega) \longrightarrow \mathcal{M}_g$

lattice

cusp of $SL(X,\omega)$ $\Leftrightarrow$ periodic slope s for $(X,|\omega|)$

$\Leftrightarrow$ cylinder system $A = (A_1, \ldots, A_n)$ +

fundamental twist τ_A, $D\tau_A \in SL(X,\omega)$

parabolic

Thurston's multi curve systems

Every Teichmüller curve V can be specified by a pair of topological multicurves (A_i), (B_j).

Gives $\langle D\tau_A, D\tau_B\rangle = \Gamma \subset SL(X,\omega)$

Usually of infinite index!

Modular symbols for V organize all the curves systems encoding V.

Thurston's multi curve systems

Every Teichmüller curve V can be specified by a pair of topological multicurves (A_i), (B_j).

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Usually of infinite index!

Theorem: *There is a natural inclusion*

$$\mathbb{S}^1(V) \longrightarrow \mathcal{ML}_g \times \mathcal{ML}_g / \mathrm{Mod}_g$$

whose image is the set of all (A,B) specifying V.

Geometry **Topology**

J = i(A_i,B_j)

modular symbol

$$(X,\omega) + \gamma \longrightarrow \left(\sum a_i \cdot A_i, \sum b_j \cdot B_j\right)$$

multicurves

$$\downarrow$$

$$Q = \begin{pmatrix} 0 & m_A J \\ m_B J^t & 0 \end{pmatrix}$$

$$\downarrow$$

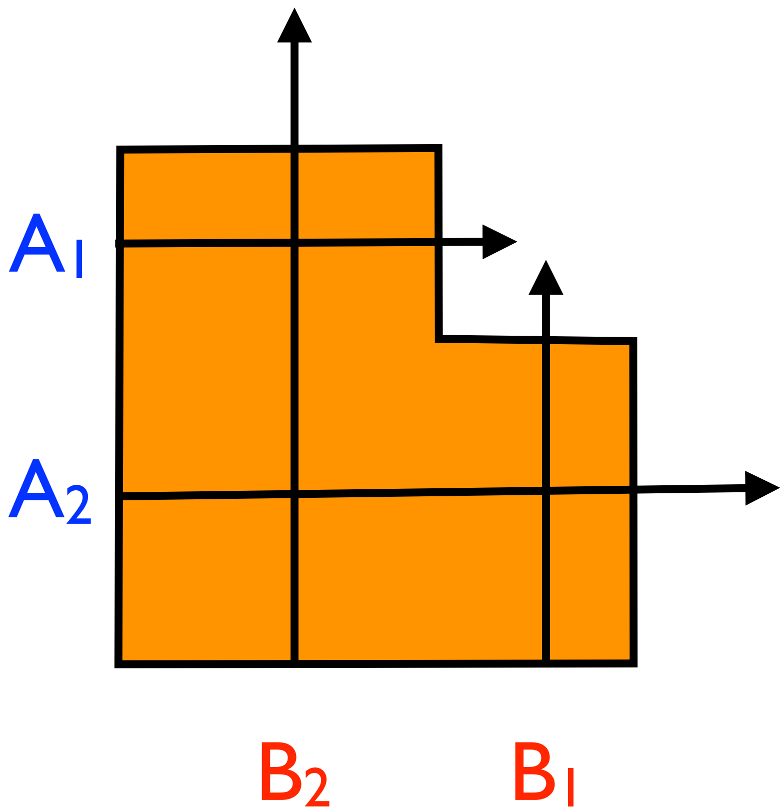
heights

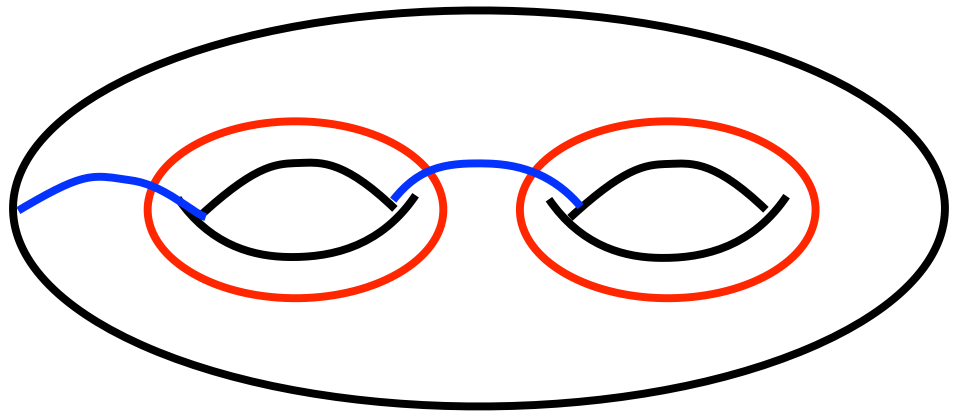
$$Q \begin{pmatrix} h_A \\ h_B \end{pmatrix} = \mu \begin{pmatrix} h_A \\ h_B \end{pmatrix}$$

$$(Y,\eta) + [0,\infty] \longleftarrow$$

$$\uparrow \ \mathrm{SL}_2(\mathbb{R})$$

Discovering the golden table





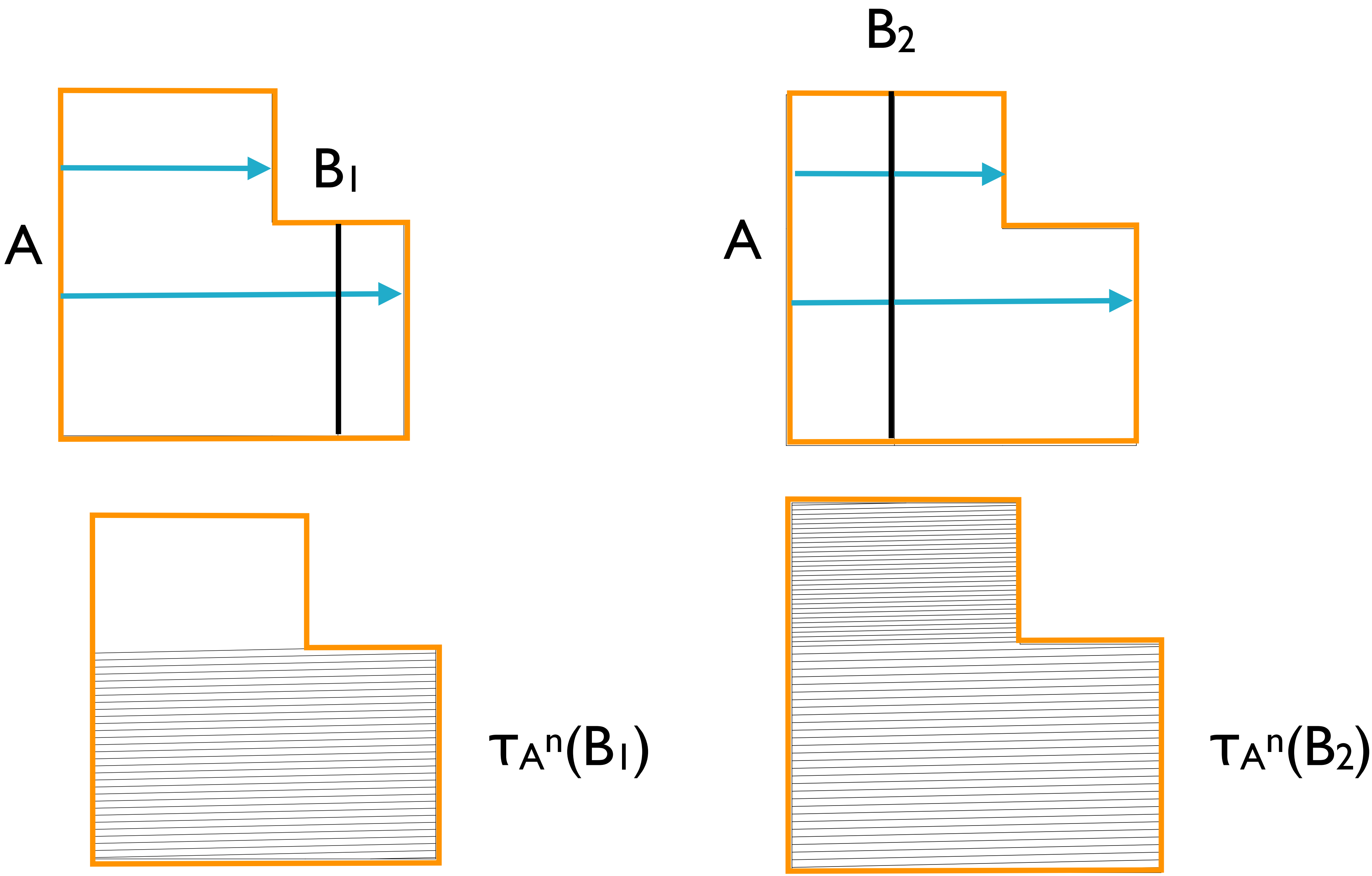
The A_4 Coxeter diagram

$$i(A_i,B_j) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$SL(X,\omega)$ is a lattice, $\simeq \Delta(2,5,\infty)$

non-arithmetic group

Twists and limit measures



Measures predicted by i(A,B)

Modular symbols to measures

We have a *continuous functor* $I : \mathbb{S}(V) \longrightarrow \mathbb{L}(V)$

category of matrices up to scale

given by $I(\gamma) = [\mathrm{mod}(A_i)\ i(A_i, B_j)]$.

Decouples as $\gamma \longrightarrow \infty$. $\sim [h(A_i)\ c(B_j)]$

$\Rightarrow$ closure of image is ω^ω union a finite set

$\Rightarrow$ limit measures form a copy of $\omega^\omega + 1$.

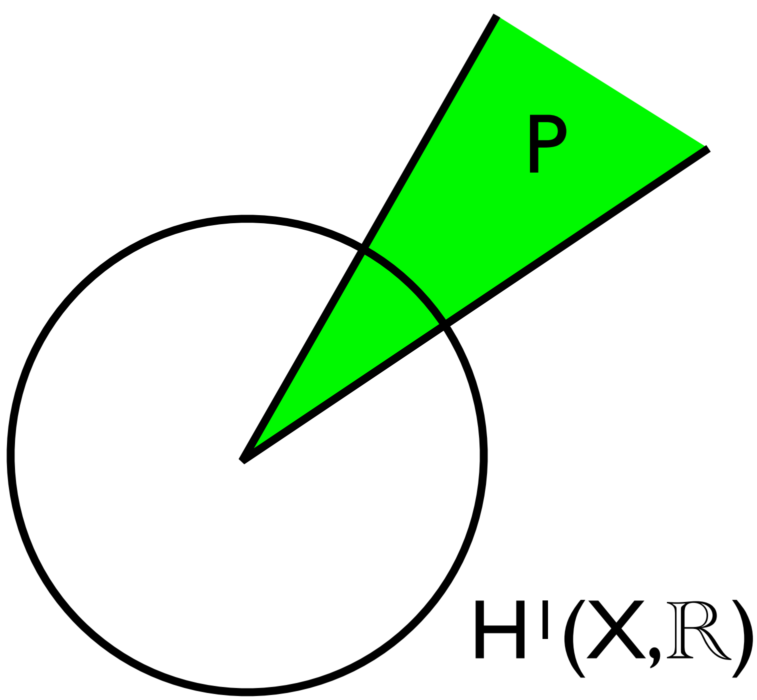
hidden multiplicative structure QED Theorem II

Theorem

If the ray generated by (X,ω) spends at least time T in a compact set K in $\mathcal{M}_g$, then the unit norm positive currents

$$[P_1(\omega)] \subset H^1(X,\mathbb{R})$$

carried by $\mathcal{F}(\omega)$ have Hodge diameter $O(\exp(-C(K)\,T))$.



$\Rightarrow$ *decoupling*

Square-tiled case

Sometimes M_s is $\omega^\omega + 1$, sometimes it is one point!

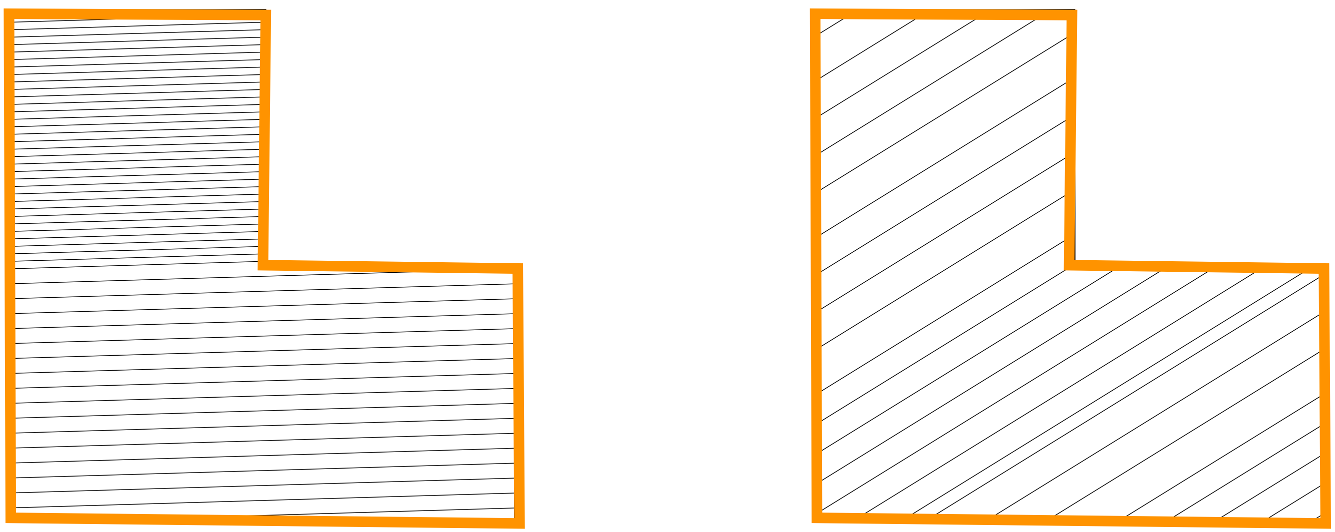
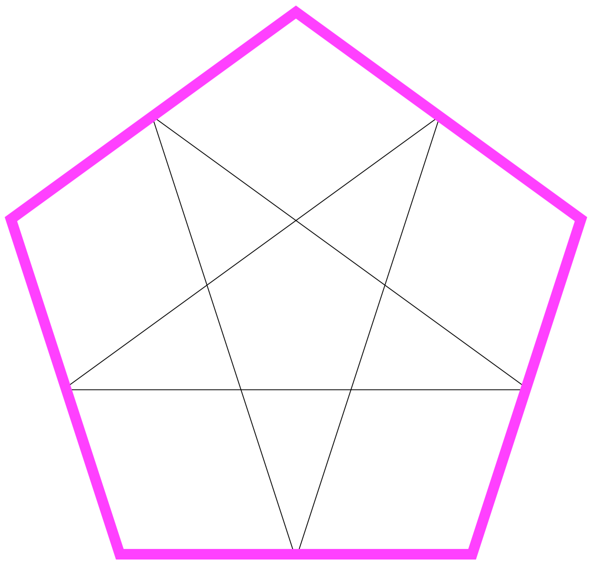


Figure 4. Periodic geodesics near slopes $s = 0$ and $s = 1$.

Slopes, lengths and heights



s

Theorem 1

The periodic slopes coincide with $\mathbb{Q}(\sqrt{5})$s, and $\log L(xs) = O(h(x)^2)$.

$$h(p/q + r/s\,\sqrt{5}) \simeq \log \max(|p|,|q|,|r|,|s|) \geq 0$$

Method: *descent,* using a new *height* on $\mathbb{P}^1(K)$

Curves on a Hilbert modular surface

cf. M, Möller-Viehweg

K = real quadratic field

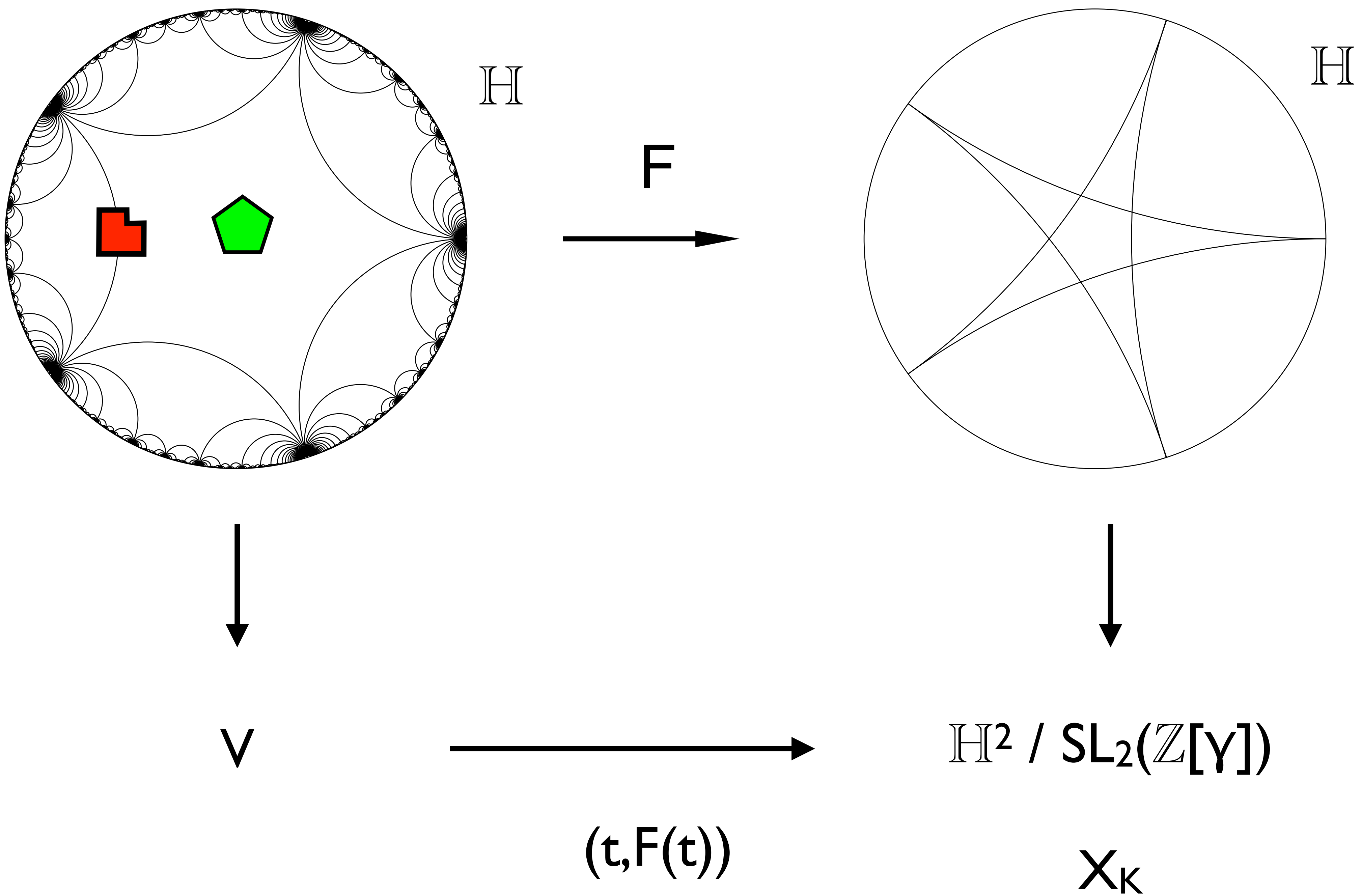
$$X_K = (\mathbb{H} \times \mathbb{H}) / \mathrm{SL}(\mathcal{O} \oplus \mathcal{O}^\vee)$$

$V = \mathbb{H}/\Gamma \hookrightarrow X_K$ geodesic curve

Theorem 1

Either V is a Shimura curve, or the cusps of V coincide with $\mathbb{P}^1(K)$ *and satisfy quadratic height bounds.*

Relation to the pentagon



Symmetries of Teichmüller curves

Theorem (M,Möller)

SL(X,ω) *a lattice with trace field* K ⇒

A=Jac(X) *admits real multiplication by* K

(a factor of) $K \subset End(A) \otimes \mathbb{Q}$

Idea: $$g + g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \mathrm{Tr}(g) \cdot I$$

$$(\phi + \phi^{-1})^* \omega = (\mathrm{Tr}\, \phi) \omega$$

The projective line $\mathbb{P}^1_A(K)$

$$K \subset \mathrm{End}(A) \otimes \mathbb{Q}$$

$$H_1(A, \mathbb{Q}) \cong K^2$$

$\mathbb{P}^1_A(K)$ = space of K-lines in $H_1(A, \mathbb{Q})$

Classical height on $\mathbb{P}^n$(K)

$$H(x) = H(x_0 : x_1 : \cdots : x_n) = \prod_v \max_i |x_i|_v.$$

comparable to

$$\widetilde{H}(x) = \inf_a \prod_{v|\infty} \max_i |a_i|_v, \qquad [a_0 : \cdots : a_n] = [x].$$

(a_i are integers)

only requires knowledge of integers and infinite places

Height $H_A(x)$ on $\mathbb{P}^1_A(K)$

$$H_A(x) = \inf_C \prod_{v|\infty} |C|_v$$

Hodge norm at v

$x \in \mathbb{P}_A{}^1(K)$

$C \in H_1(X, \mathbb{Z})$

$x = [K \cdot C]$

$$|C|_v = \left| \int_C \omega_v \right|^{1/g}$$

Why a height?

$$H_A(x) = \inf_C \prod_{v|\infty} |C|_v$$

$$\widetilde{H}(x) = \inf_a \prod_{v|\infty} \max_i |a_i|_v$$

Theorem. Given a linear isomorphism

$$\iota : \mathbb{P}^1_A(K) \to \mathbb{P}^1(K)$$

we have $$H(\iota(x)) \asymp H_A(x).$$

Case of a torus

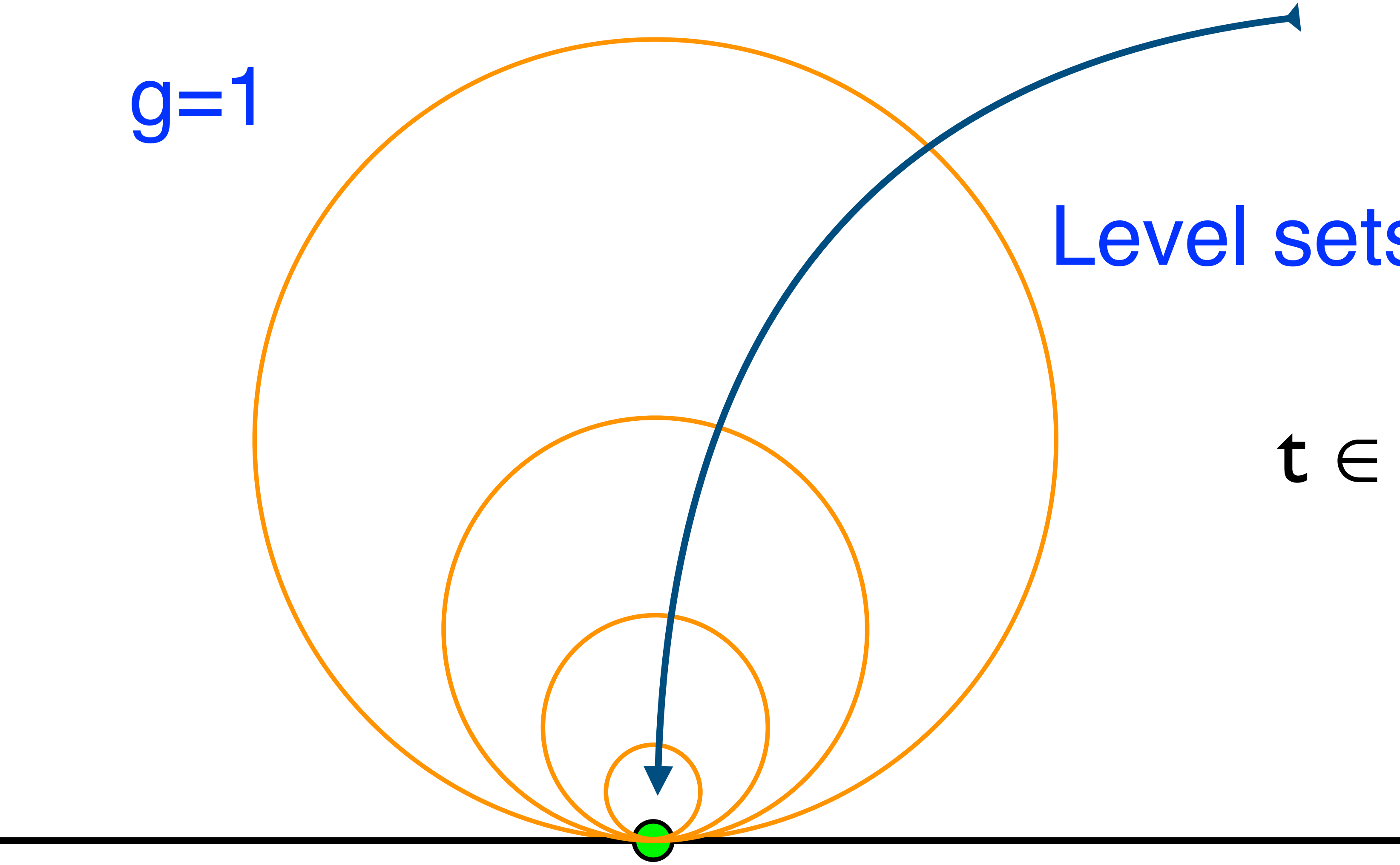
$$A = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau \qquad H_1(A, \mathbb{Z}) \cong \mathbb{Z}^2$$

$$K = \mathbb{Q}$$

$$\|C\|_A^2 = \left| \int_C \omega \right|^2 = \frac{|a + b\tau|^2}{\operatorname{Im} \tau}$$

Hodge norm

$H_\tau(x)$ = length of geodesic with slope $x = a/b$



Level sets of height

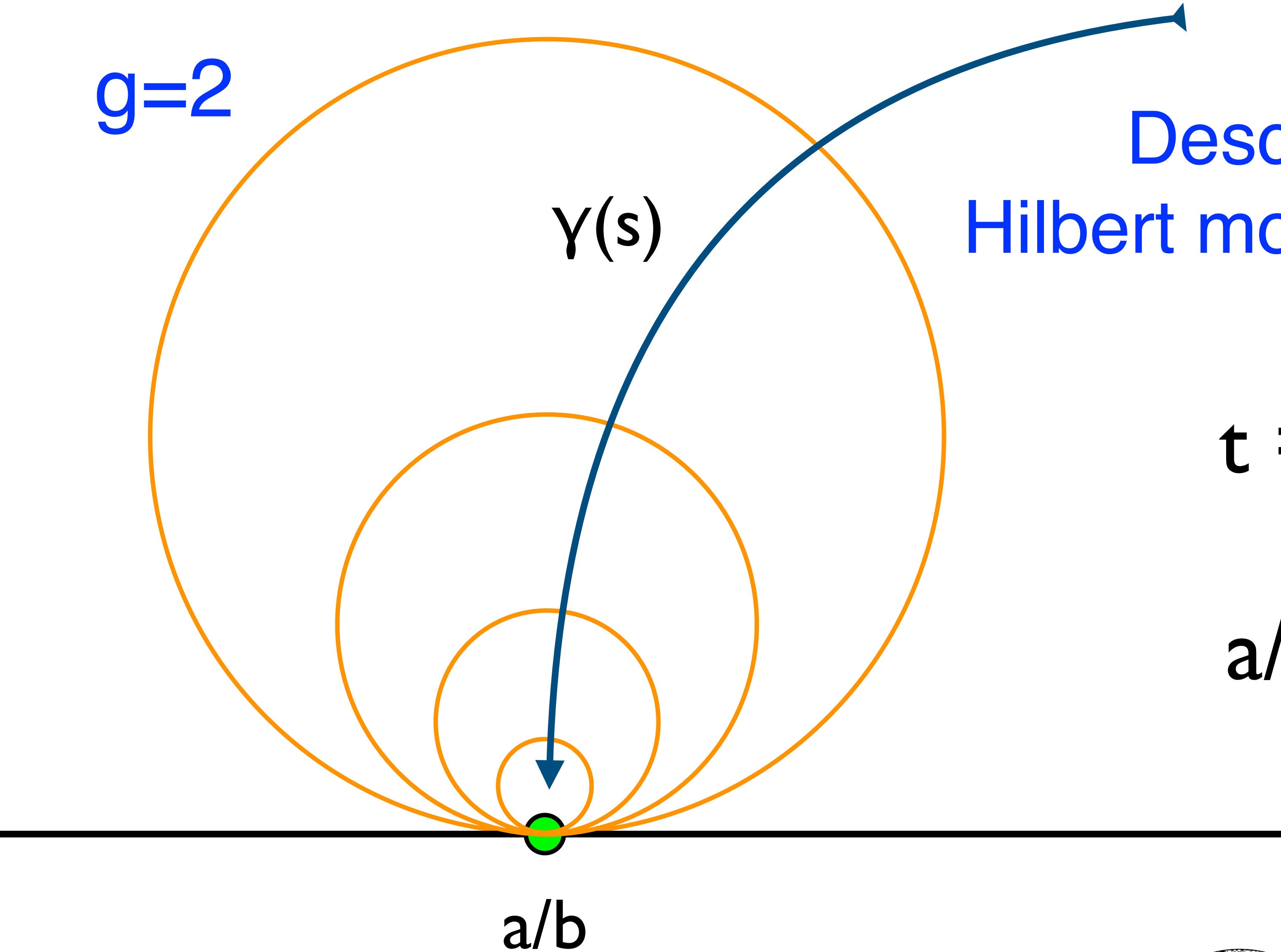
$\mathbf{t} \in \mathbb{H}$

a/b

$H_t(a/b)$

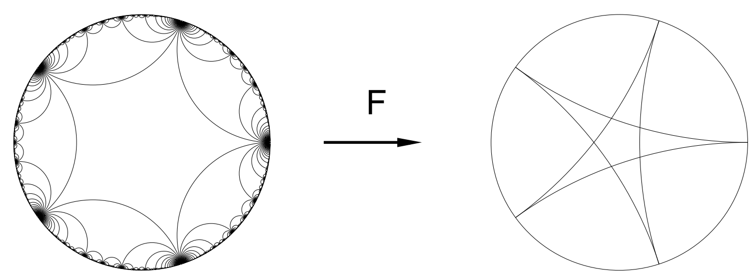
decreases like exp(-s) along geodesic rays descending to a/b g=2

Descent on a Hilbert modular surface



$$t = \gamma(s) \in \mathbb{H}$$

$$a/b \in \mathbb{Q}(\sqrt{D})$$



$H_\tau(a/b)$ $\qquad$ $\tau = (t, F(t))$

$$A_\tau = \mathbb{C}^2 / \mathcal{O} \oplus \tau\mathcal{O}^\vee$$

To show a/b is a cusp

$$H_\tau(a/b) \sim (t \text{ term}) \times (F(t) \text{ term})$$

$$\leq \exp(-s)\exp(|F'|\, s)$$

When t lies over V_{thick} :

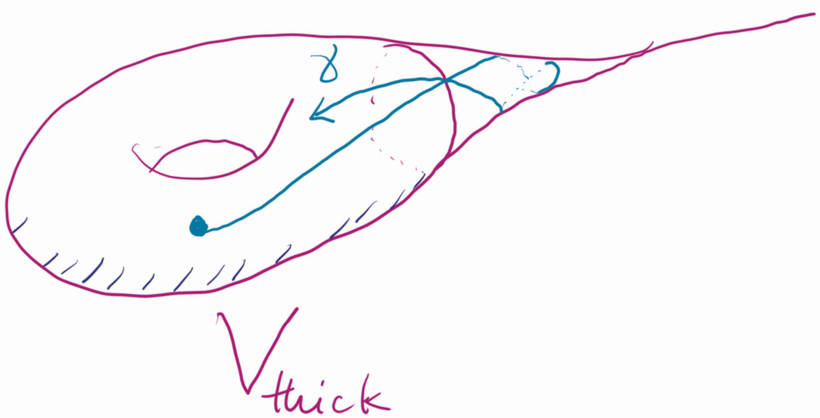
$H_\tau(a/b) \geq 1$

$|F'(t)| < \delta < 1$

So γ spends only a finite amount of time over V_{thick}

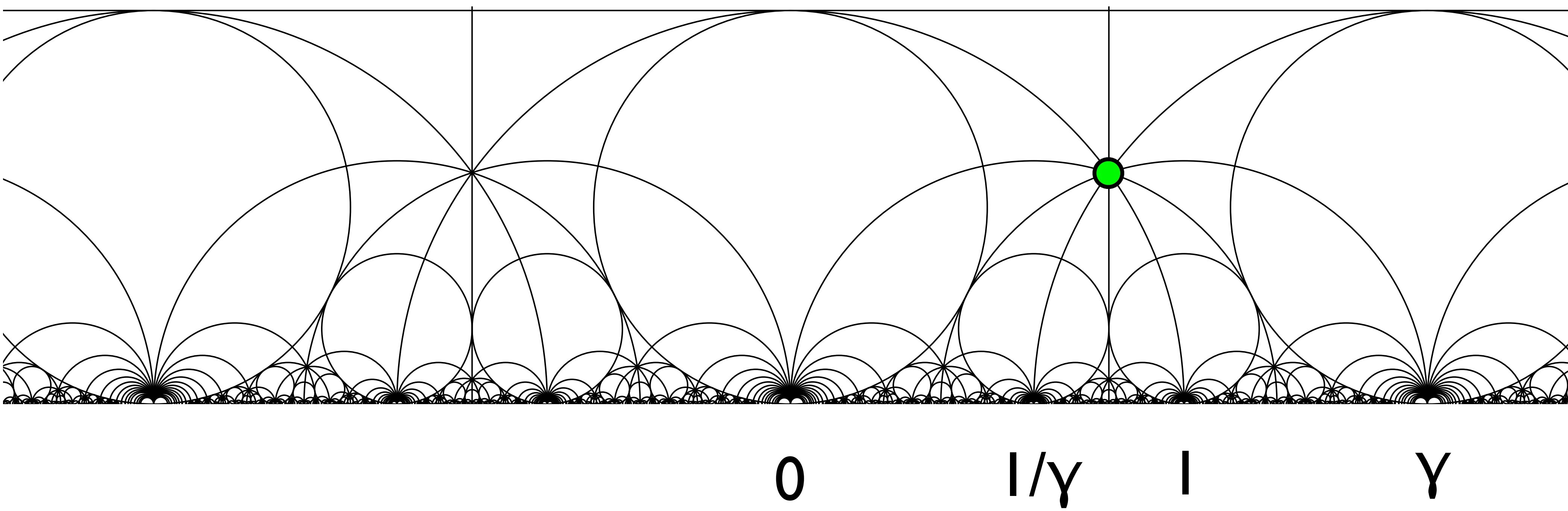
$\Rightarrow$

a/b is a cusp



Cor: Triangle groups

$$\Gamma = \langle z \mapsto -1/z \quad \text{and} \quad z \mapsto z + \gamma \rangle$$



The cusps of the (2,5,∞) triangle group Γ coincide with K.

Cor: Continued fractions

Every $s \in \mathbb{Q}(\gamma)$ can be expanded as a finite golden continued fraction,

$$s = [a_1, a_2, a_3, \ldots, a_N] = a_1\gamma + \cfrac{1}{a_2\gamma + \cfrac{1}{a_3\gamma + \cdots \cfrac{1}{a_N\gamma}}}$$

with $a_i \in \mathbb{Z}$.

Height bounds: length N and a_i are O(1+h(x)) .

Theorem

The cusps of the (p,q,∞) triangle group coincide with $\mathbb{P}^1(K)$ whenever $\deg(K/\mathbb{Q}) = 1$ or 2.

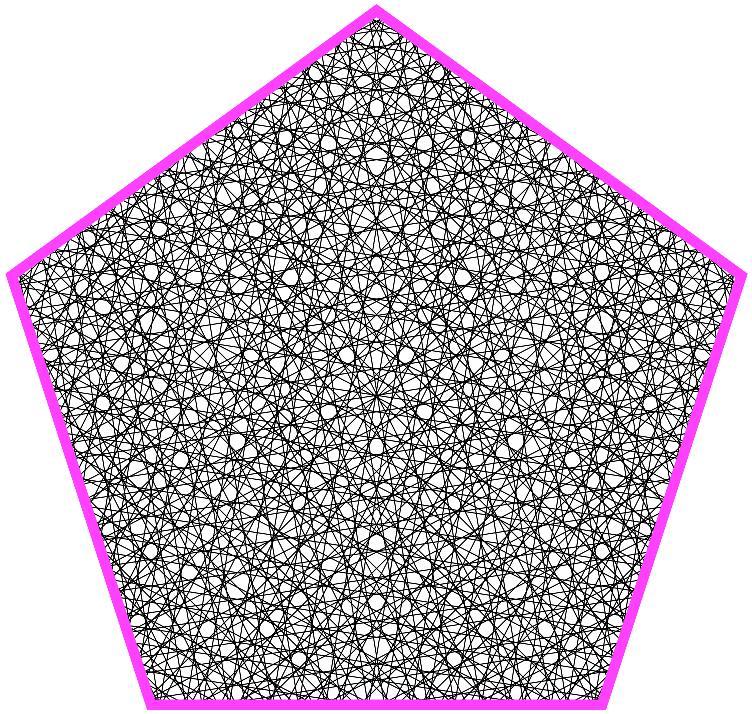
non arithmetic

Open problem

Converse? For triangle groups and for Veech groups?

$(2,q,\infty)$ known

Cor: Billiards in a pentagon

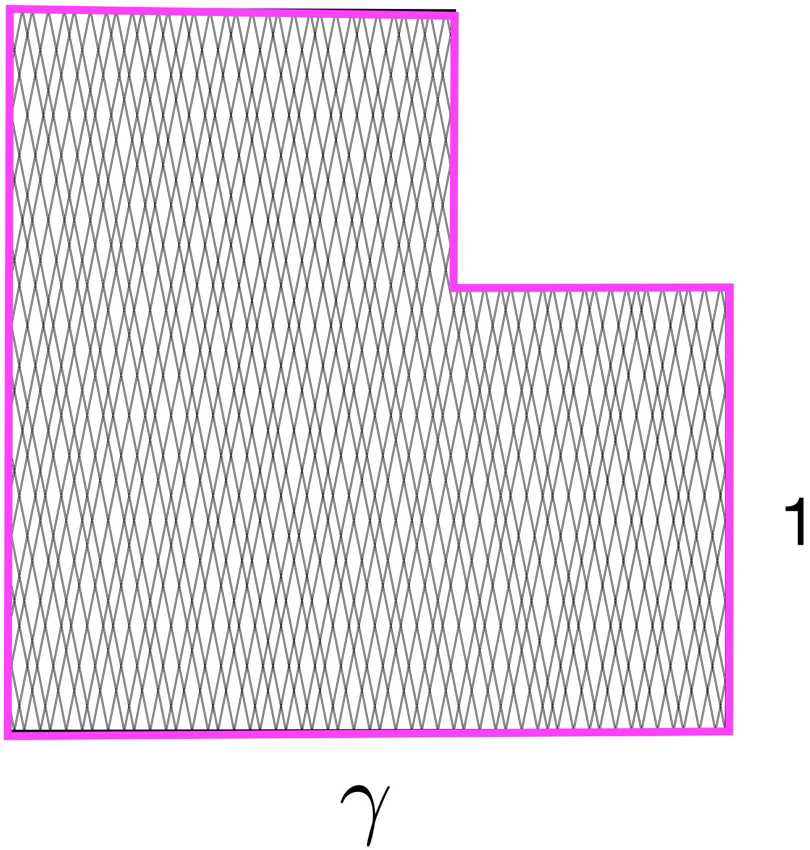


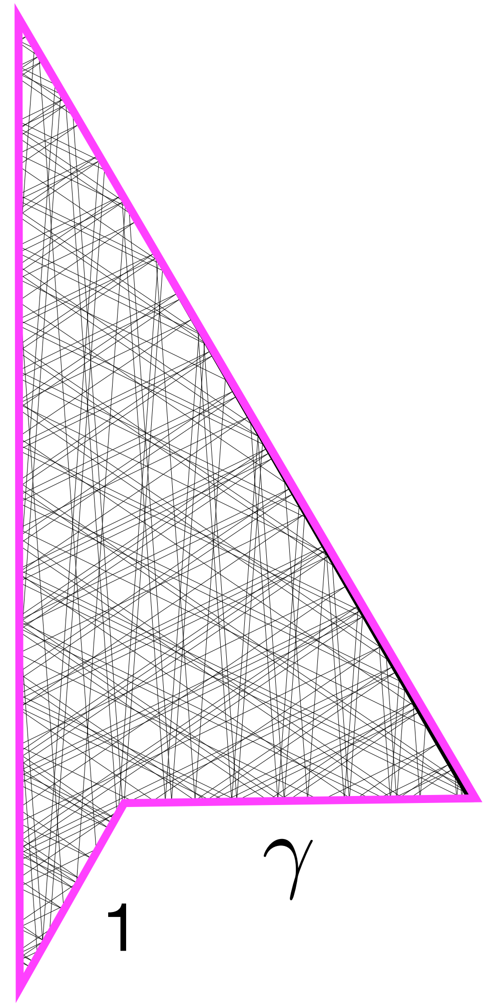
$s = \tan(2\pi/5)$

The periodic slopes coincide with $\mathbb{Q}(\sqrt{5})s$*, and*

$\log L(xs) = O(h(x)^2)$.

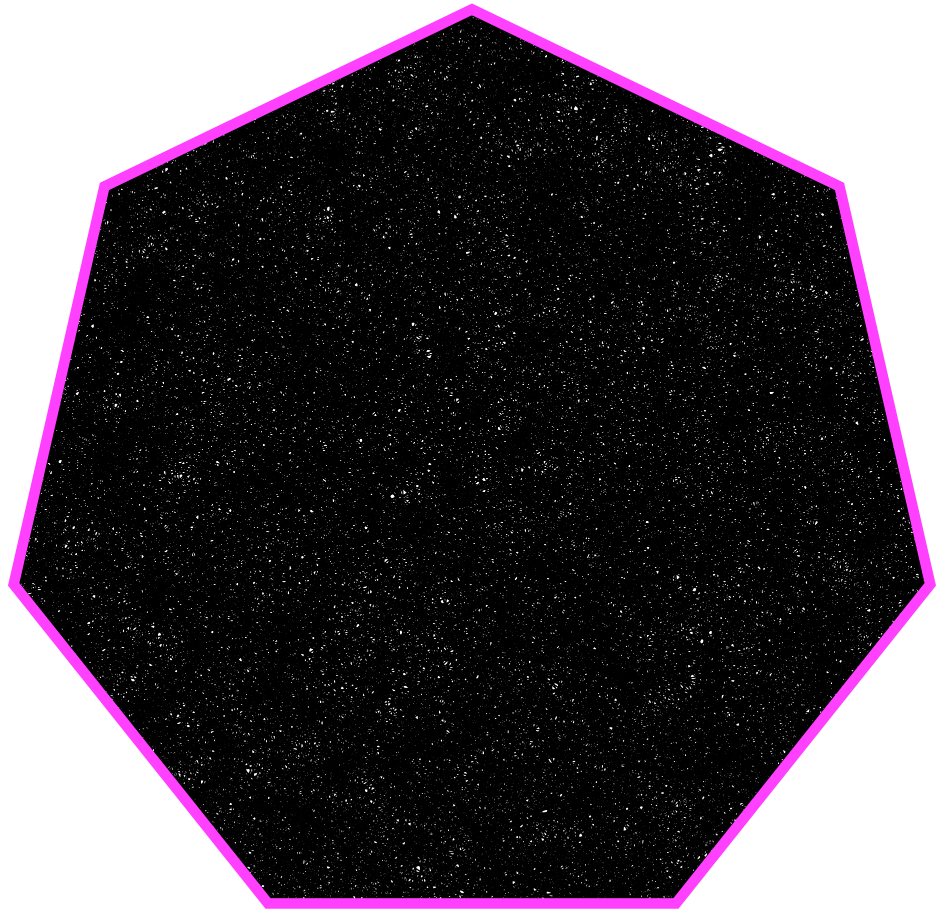
Applies to all families of optimal billiards





…since these are quadratic: *Eskin - Filip - Wright*

Open problem



s = tan(2π/7)

K = $\mathbb{Q}$(cos(2π/7))
(cubic)

In a regular heptagon,
(i) characterize the periodic slopes, and
(ii) bound L(x s), x in K.

Shown: L(s)=7, L(2 s) = 2190

References

Preprints, 2019/2020

Teichmüller dynamics and unique ergodicity via currents and Hodge theory

Modular symbols for Teichmüller curves

Billiards, heights, and the arithmetic of non-arithmetic groups
in preparation

math.harvard.edu/~ctm/papers