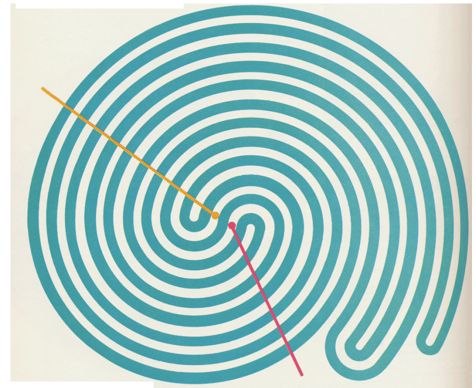
Manifolds, topology and dynamics

Milnor's work and subsequent developments

C. McMullen

Can you knot a circle in the plane?

No



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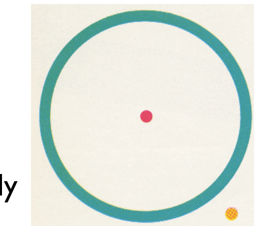
topologically



Can you knot a sphere in space?

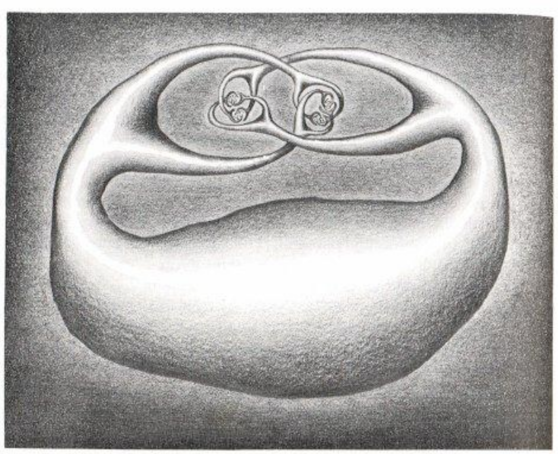
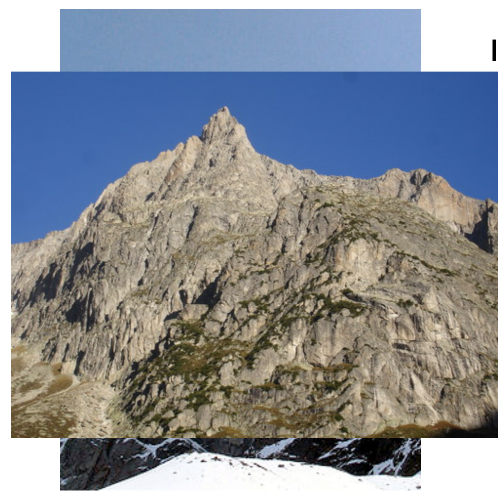


FIG. 4–11. The Alexander horned sphere.

Yes
1924
J.W. Alexander

A knotted topological S^2 in S^3

No
for smooth spheres



Aiguille du Drus (Chamonix)

1926 deRham and Alexander exchange ice axes

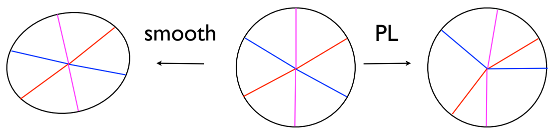
1933 deRham runs into Alexander and Whitney Weisshorn, Valais

1956 Whitney and Milnor climb Ixtaccihuatl

1966 deRham and Milnor climb south ridge of Stockhorn

Smooth spheres (Milnor, 1956)

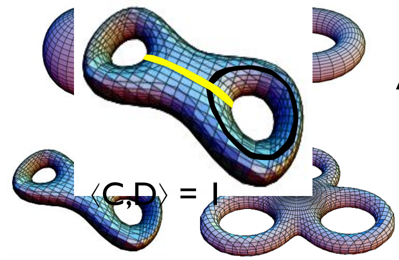
There exists a smooth manifold M^7 that is homeomorphic but not diffeomorphic to S^7.



$\Rightarrow$ *Smooth Poincaré conjecture is false in high dimensions:*

$\pi_1(M^n)=\pi_1(S^n)$ and $H_i(M^n)= H_i(S^n)$
does not guarantee M^n is diffeomorphic to S^n.

Classification of Manifolds



Surfaces

A manifold of dimension 2 is determined by the inner product space $H^1(M^2,\mathbb{Z}/2)$.

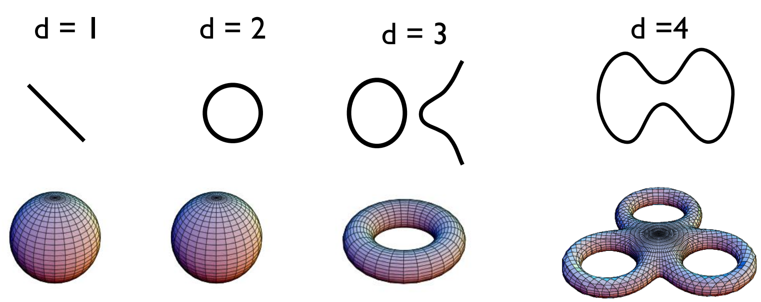
4 manifolds (Milnor, 1958)

An oriented simply-connected 4-manifold is determined up to homotopy equivalence by the inner product space $H^2(M^4,\mathbb{Z})$.

Where do manifolds come from?

$M^2(d) \subset \mathbb{C}^2$ given by polynomial of degree d, e.g. $x^d+y^d = 1$

(*then compactified*)



4-manifolds from polynomials

$M^4(d) \subset \mathbb{C}^3$ given by polynomial of degree d, e.g. $x^d+y^d+z^d = 1$

	d = 1	d = 2	d = 3	d =4
M^4	$\mathbb{C}P^2$	$S^2 \times S^2$	$\mathbb{C}P^2 \# 6\, \overline{\mathbb{C}P^2}$	K3 Surface
$H^2(M^4)$	[1]	$H=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$[1]\oplus 6[-1]$	$2E_8 \oplus 3H$
	I_1	$II_{1,1}$	$I_{1,6}$	$II_{3,19}$

Classification of simply-connected 4-manifolds

(Freedman, Donaldson, early 1980s)

Topological

Any inner product space can arise as $H^2(M^4,\mathbb{Z})$, in at most 2 ways.

Smooth

There exist topological M^4 with infinitely many smooth structures.

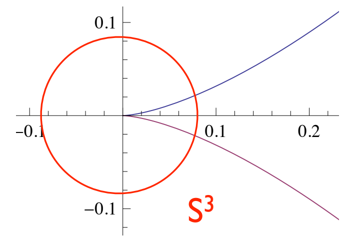
Conjecture

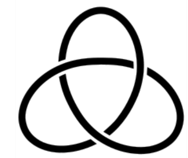
Connect sums of $\mathbb{C}P^2$, $S^2 \times S^2$, and K3 surfaces give all the homotopy types for smooth M^4.

Singularities and Knots

$\mathbb{C}^2$

$V : y^2 = x^3$





$K^1 = S^3 \cap V$ (*Milnor link*) is an ordinary S^1

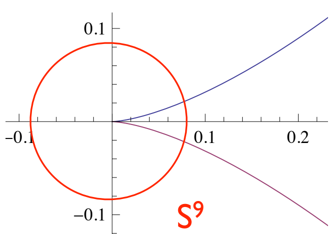
d=1: Knottedness detects singularity

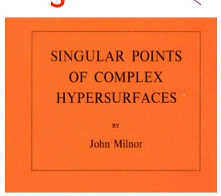
d=2 (Milnor, 1975): $K^3 = G^3/\Gamma$ is always a geometric 3-manifold.

Singularities and Spheres

$\mathbb{C}^5$

$V: \ y^2 = x_1^2 + x_2^2 + x_3^3 + x_4^5$



(see: 

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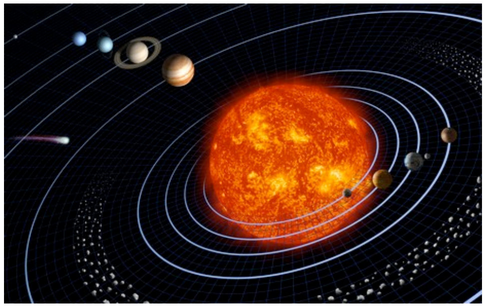
The Milnor link

$K = S^9 \cap V$

is an exotic S^7

Dynamical systems

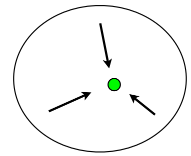


Poincaré

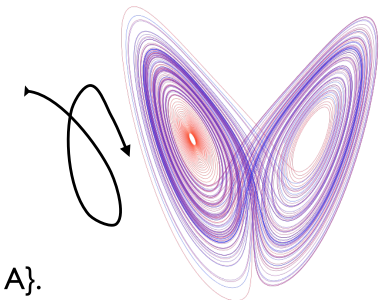
$f : M \to M$ continuous, smooth, analytic...

Study evolution of orbits: $x \to f(x) \to \ldots \to f^n(x)$

Attractors



attracting point



strange attractor

$A \subset M$ closed. $B(A) = \{x : f^n(x) \to A\}$.

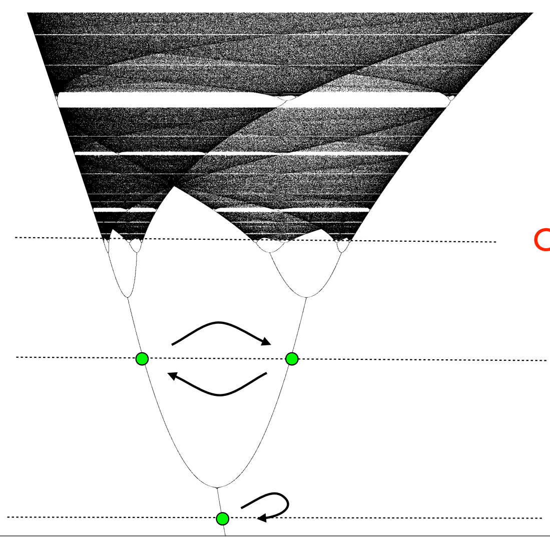
Milnor, 1985:

$A \subset M$ is an *attractor* if $B(A)$ has positive measure, and $B(E)$ has less measure for any smaller $E \subset A$.

Experiment: $f(x) = x^2 - c$ $M = [-c,c]$

c in [0,2]

Attractor A_c

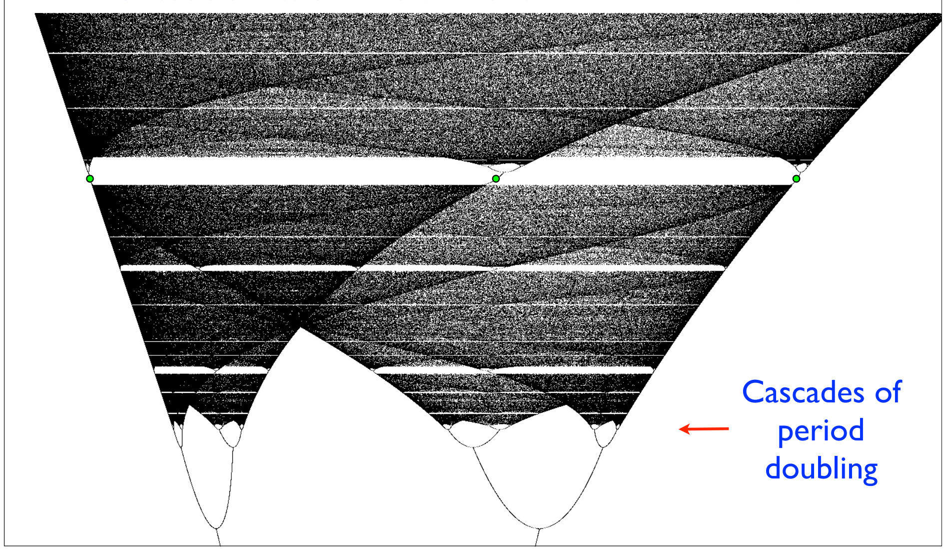


Onset of chaos

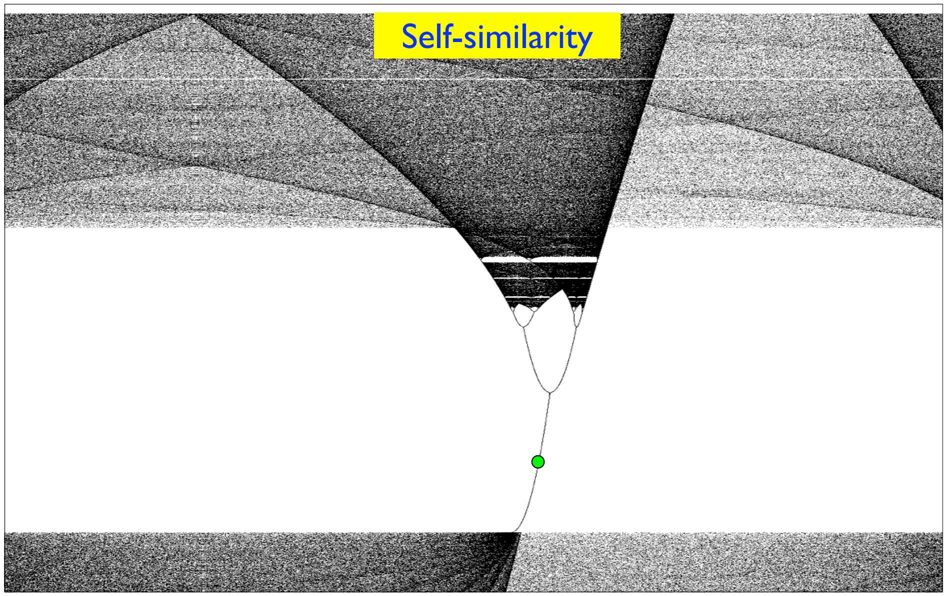
c

c

Chaotic behavior   Islands of calm   Phase transitons

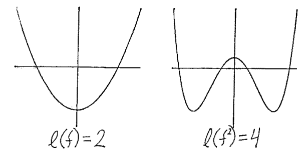


Cascades of period doubling

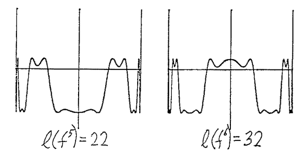


Topological maps $f : [a,b] \to [a,b]$

Milnor - Thurston 1977-83



Shows each topological f has an smooth structure.



* Kneading invariant $\varepsilon_k = \text{sign } f^k(0)$
* Topological dynamics
* Lap numbers / entropy
* Zeta function/fixed pts of f^n

$$\zeta(t) = \exp \sum [\#\text{ fixed pts of } f^n]\, t^n/n$$

$$D(t) = \sum \varepsilon_1 \varepsilon_2 \ldots \varepsilon_n t^n$$

$$\zeta(t) = 1/D(t)$$

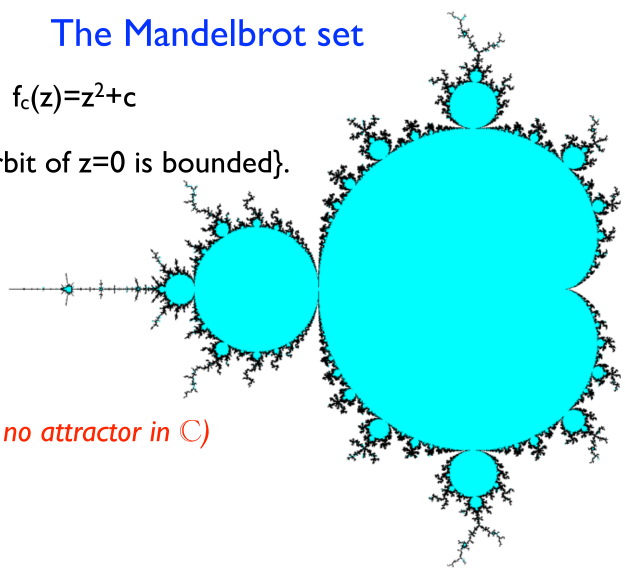
Explains self-similarity

The Mandelbrot set

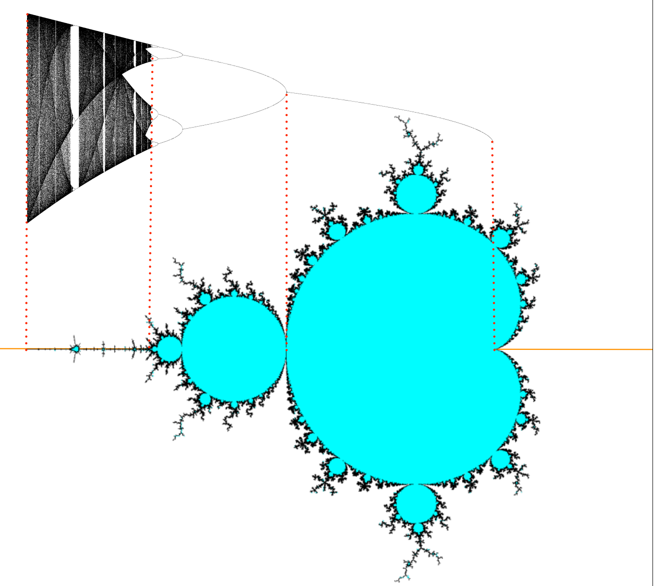
$f_c(z)=z^2+c$

$M=\{c : \text{orbit of } z=0 \text{ is bounded}\}$.

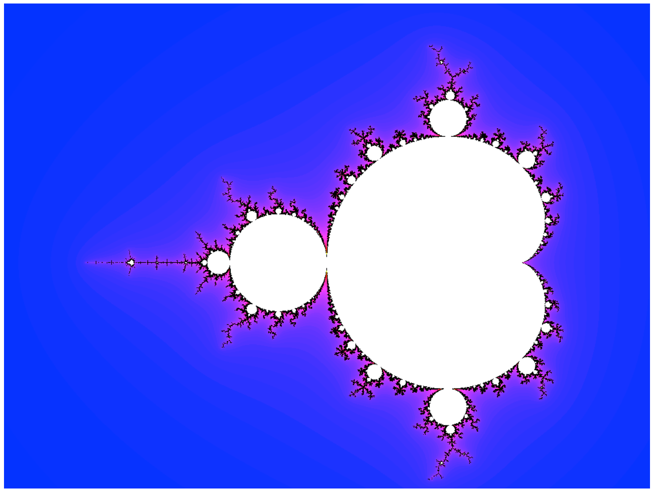
(otherwise no attractor in $\mathbb{C}$)



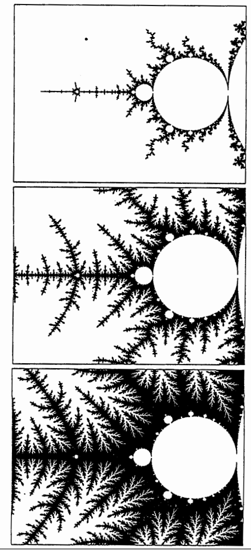
Real points in M



Milnor, 1986: Self-similarity of M



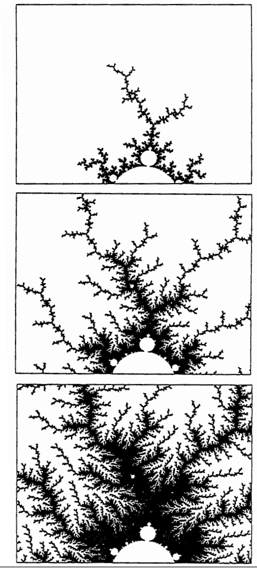
Renormalization

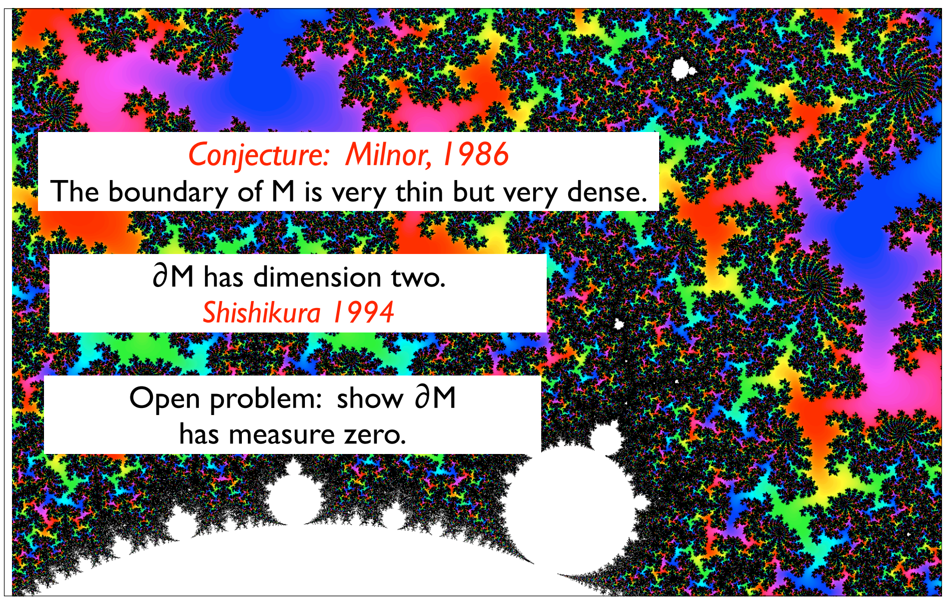


Conjecture: Milnor, 1986

The magnifications of M near any fixed-point of renormalization c are self-similar and fill the plane.

Theorem: Lyubich, 1999 for c in $\mathbb{R}$.





Other Frontiers in Complex Dynamics

* cubic polynomials, quadratic maps $f : \mathbb{P}^1 \to \mathbb{P}^1$

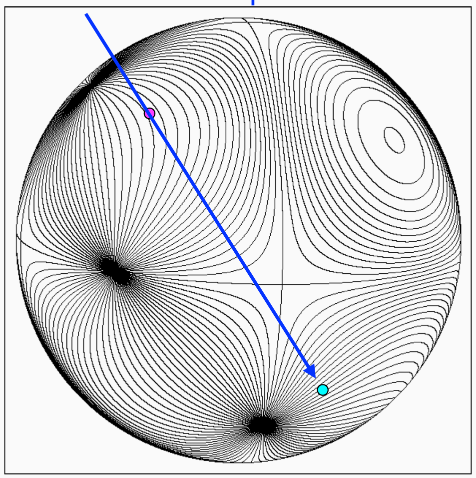
* polynomial maps $f : \mathbb{C}^2 \to \mathbb{C}^2$

* higher degree maps $f : \mathbb{P}^2 \to \mathbb{P}^2$

* automorphisms of complex surfaces $f: M \to M$

Bonifant-Dabija-Milnor 2007:

Maps on $\mathbb{P}^2$ with invariant elliptic curves

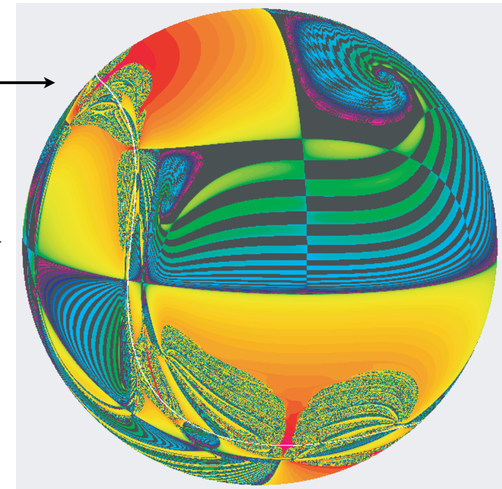


foliation by cubic curves

Attracting behavior of $C \cup L_i$'s

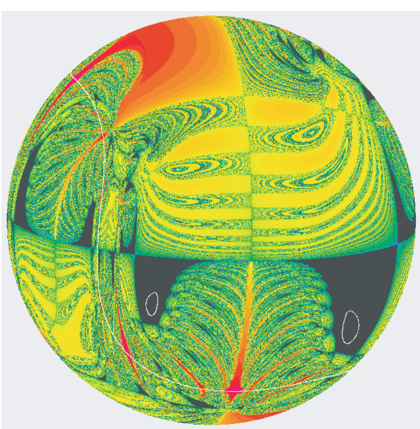
Fermat cubic C →

Equator L →

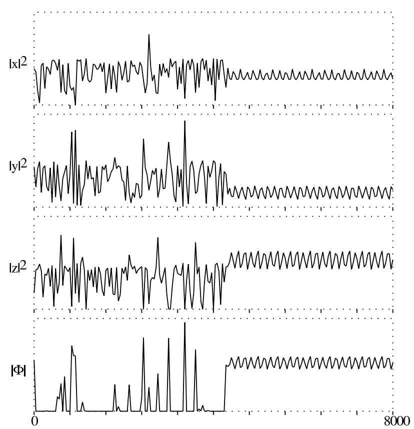


illustrations from BDM 2007

Attracting Herman rings?



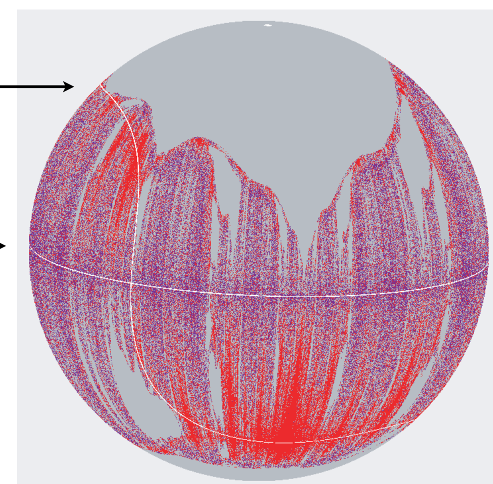
Typical orbit over $\mathbb{C}$



Intermingled basins

Fermat cubic C →

Equator L →



basins of L, C have positive measure

and same closure!

Automorphisms of projective surfaces M

Cantat, 1999

Entropy h(f : M → M) > 0 ⇒

M is a complex torus, blowup of $\mathbb{C}P^2$, or K3 surface.

Minimum entropy?

Torus: h(f) = log(λ_4 = 1.722...).

Blowup of $\mathbb{C}P^2$ (Bedford-Kim, 2006): h(f) = log(λ_{10}=1.176...).

K3 Surface (M, 2011): h(f) = log(λ_{10}).

- Hodge theory and quadratic forms over number fields and local rings to construct

$f \circlearrowright H^2(M,\mathbb{Z})\otimes\mathbb{C} = H^{2,0}\oplus H^{1,1}\oplus H^{0,2}$