Billiards, quadrilaterals and moduli spaces

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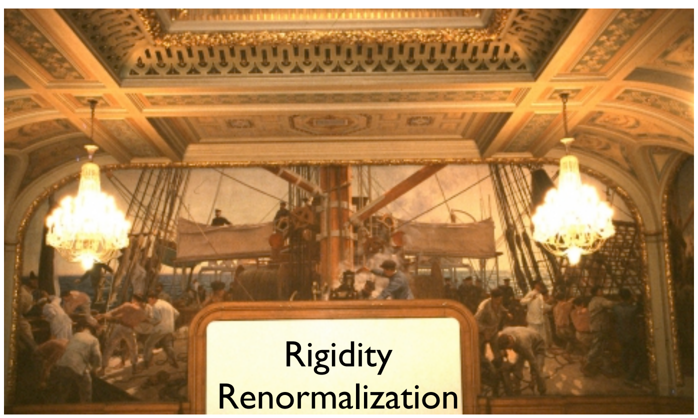
coworkers Eskin, Mukamel and Wright

Dynamics on moduli spaces

Polygonal billiards

We are still in the Age of Discovery



Dynamics on moduli spaces

- There exists a totally geodesic complex surface F in $\mathcal{M}_{g,n}$.
- Billiards in suitable darts have optimal dynamics.

A Little History

$$\int \frac{dx}{Q(x)^{1/d}}$$

$Q(x)$ a polynomial

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$$

$$\int \frac{dx}{\sqrt[3]{1-x^3}} =$$

$$\frac{3\sqrt[3]{\frac{x-1}{1+\sqrt[3]{-1}}+1}\sqrt[3]{\frac{x-1}{1-(-1)^{2/3}}+1}(x-1)F_1\left(\frac{2}{3};\frac{1}{3},\frac{1}{3};\frac{5}{3};-\frac{x-1}{1-(-1)^{2/3}},-\frac{x-1}{1+\sqrt[3]{-1}}\right)}{2\sqrt[3]{1-x^3}}$$

F_1 = Appel hypergeometric function

A Little History

$$\int \frac{dx}{Q(x)^{1/d}}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$$

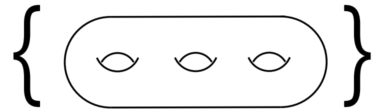
(X,ω) = (The curve $y^d = Q(x)$, the form dx/y)

periods of (X,ω) = $\{\int_C \omega\}$

→ Riemann surfaces, homology, Hodge theory, automorphic forms, ...

Moduli space

$\mathcal{M}_g$ = moduli space of Riemann surfaces X of genus g



-- a complex variety, dimension 3g-3

Teichmüller metric: every holomorphic map

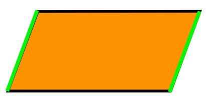
$$f : \mathbb{H}^2 \to \mathcal{M}_g$$

is distance-decreasing.

Static

How to describe X in $\mathcal{M}_g$?

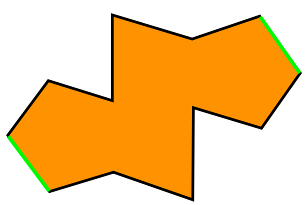
g=1: $X = \mathbb{C}/\Lambda$



g>1: X = ? *Uniformization Theorem*

Every X in $\mathcal{M}_g$ can be built from a polygon in $\mathbb{C}$

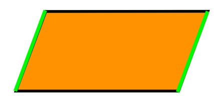
X = P / gluing
by translations



P

How to describe X in $\mathcal{M}_g$?

g=1: $X = \mathbb{C}/\Lambda$



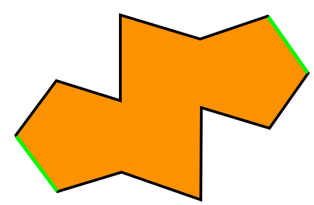
g>1: X = ? *Uniformization Theorem*

Every (X,ω) in $\Omega\mathcal{M}_g$ can be built from a polygon in $\mathbb{C}$

$\downarrow$

$\mathcal{M}_g$

$(X,\omega) = (P,dz)$ / gluing
by translations



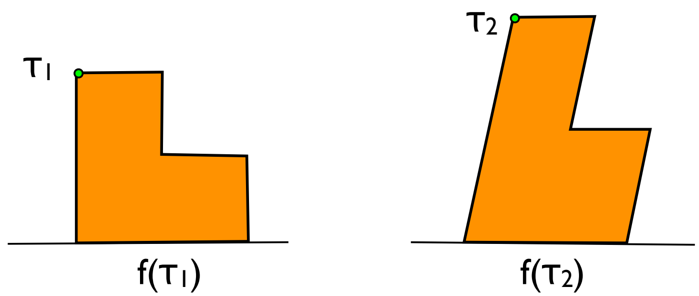
P

Moduli space $\Omega\mathcal{M}_g$

Dynamic:
$SL_2(\mathbb{R})$ acts on $\Omega\mathcal{M}_g$

Polygon for $A \cdot (X,\omega) = A \cdot$ (Polygon for (X,ω))

Complex geodesics $f : \mathbb{H} \longrightarrow \mathcal{M}_g$



Teichmüller curves

The complex geodesic generated by (X,ω) factors:

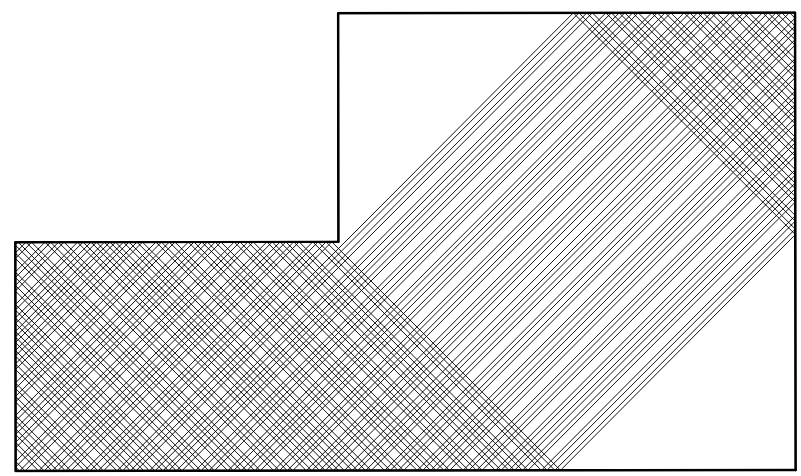
$$\begin{array}{ccc} \mathbb{H} & \longrightarrow & \mathcal{M}_g \\ & \searrow \quad \nearrow f & \\ & V = \mathbb{H} / SL(X,\omega) & \end{array}$$

stabilizer of (X,ω) in $SL_2(\mathbb{R})$

$SL(X,\omega)$ lattice $\Leftrightarrow f : V \to \mathcal{M}_g$ is an algebraic, isometrically immersed *Teichmüller curve*.

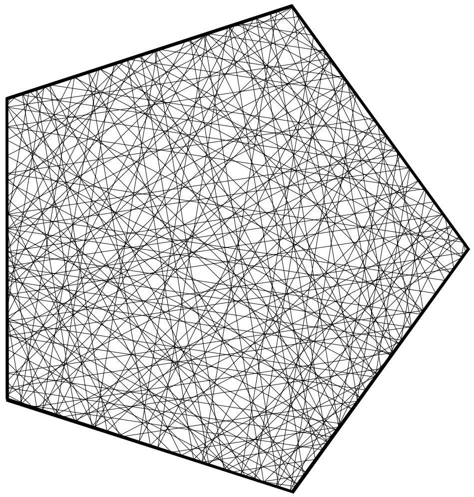
Are there any examples of Teichmüller curves?

Billiards in polygons



Neither periodic nor evenly distributed

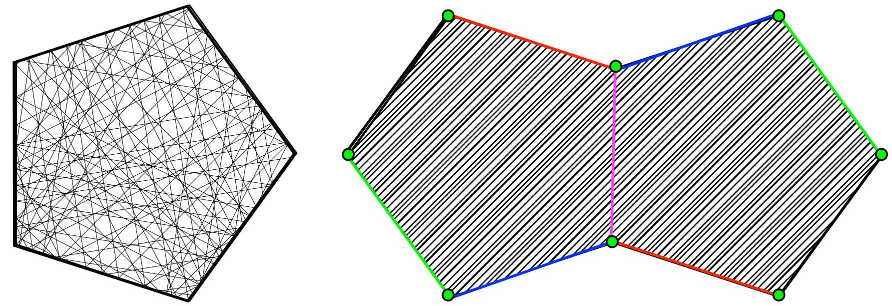
Optimal Billiards



(Veech)

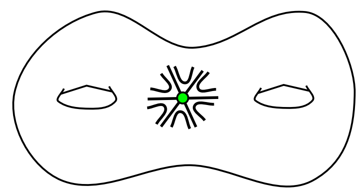
Theorem. In a regular n-gon, every billiard path is either periodic or uniformly distributed.

Billiards and Riemann surfaces



P

$(X,\omega) = P/\sim$



X has genus 2
ω has just one zero!

P is a *Lattice Polygon*

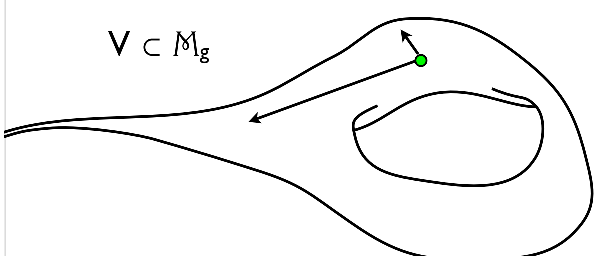
⇔ $SL(X,\omega)$ is a lattice

⇔ (X,ω) generates a Teichmüller curve

Theorem (Veech, Masur):

If P is a lattice polygon, then billiards in P is optimal.

$V \subset \mathcal{M}_g$



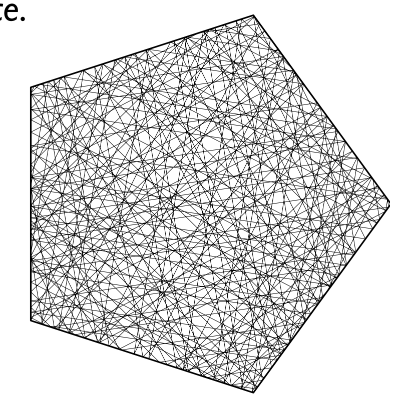
(Renormalization)

Every geodesic on V is either
recurrent or
exits a cusp

Optimal Billiards

Theorem (Veech, 1989): *Any regular polygon generates a Teichmüller curve in moduli space.*

Corollary
Billiards in a regular polygon has optimal dynamics.



Beyond Teichmüller curves?

$\mathcal{M}_g \subset \mathbb{P}^N$ is a projective variety

Almost all subvarieties $V \subset \mathcal{M}_g$ are *contracted*.

PROBLEM

Are there any *totally geodesic** subvarieties $V \subset \mathcal{M}_g$ with $\dim(V)>1$?

(*Every complex geodesic *tangent* to V is contained in V.)

The flex locus F

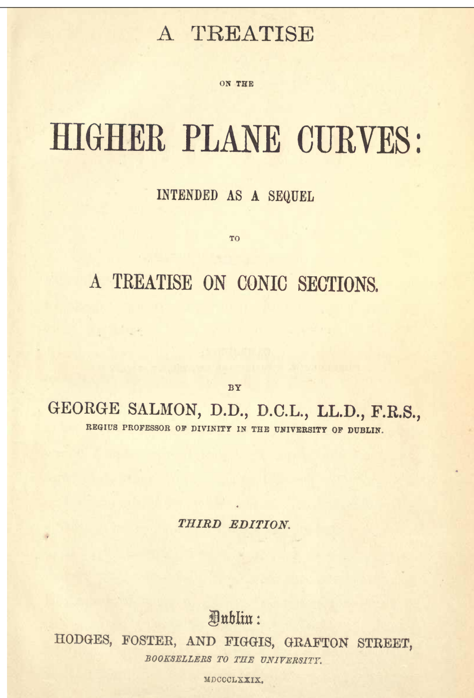
Main Theorem

There is a primitive, totally geodesic complex surface $F \subset \mathcal{M}_{1,3}$.

The universal cover T → F:
a new type of Teichmüller space?

- T is not isomorphic to any traditional $\mathcal{T}_{g,n}$.
- T is not a symmetric space (*Antonakoudis*)

1879


A TREATISE

ON THE

HIGHER PLANE CURVES:

INTENDED AS A SEQUEL

TO

A TREATISE ON CONIC SECTIONS.

BY

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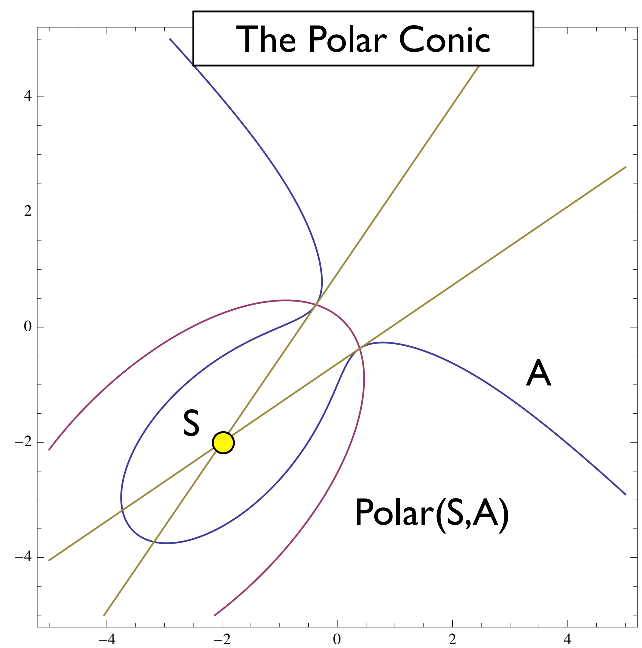
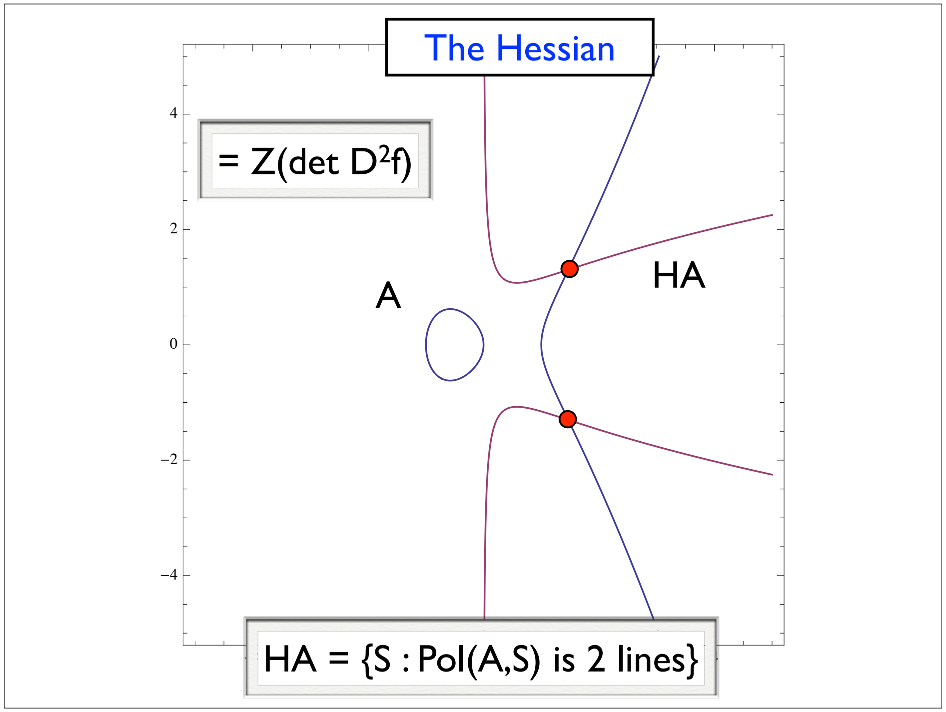
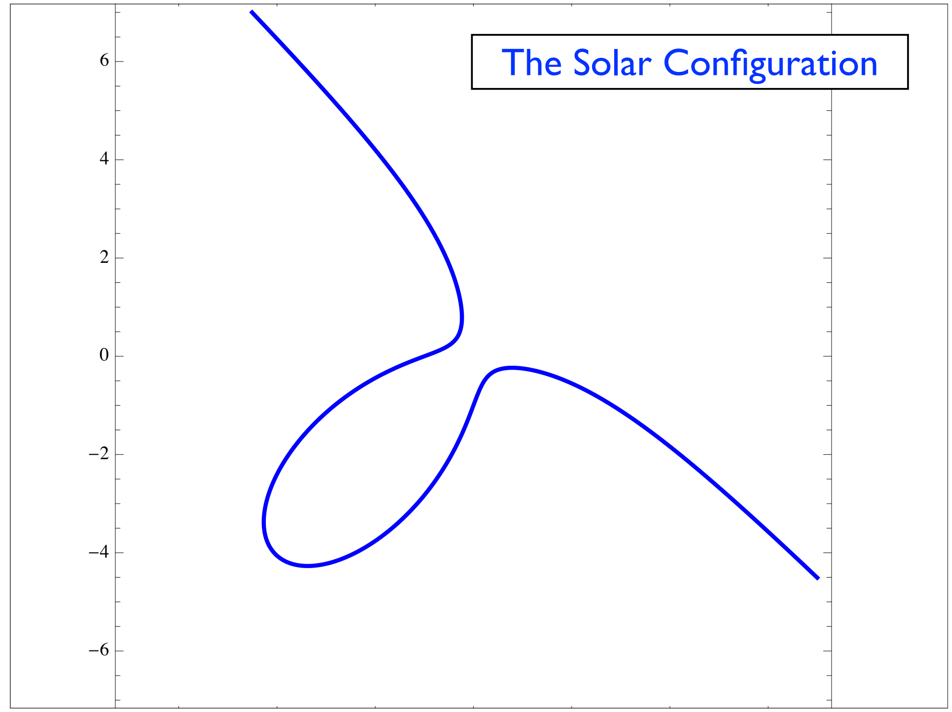
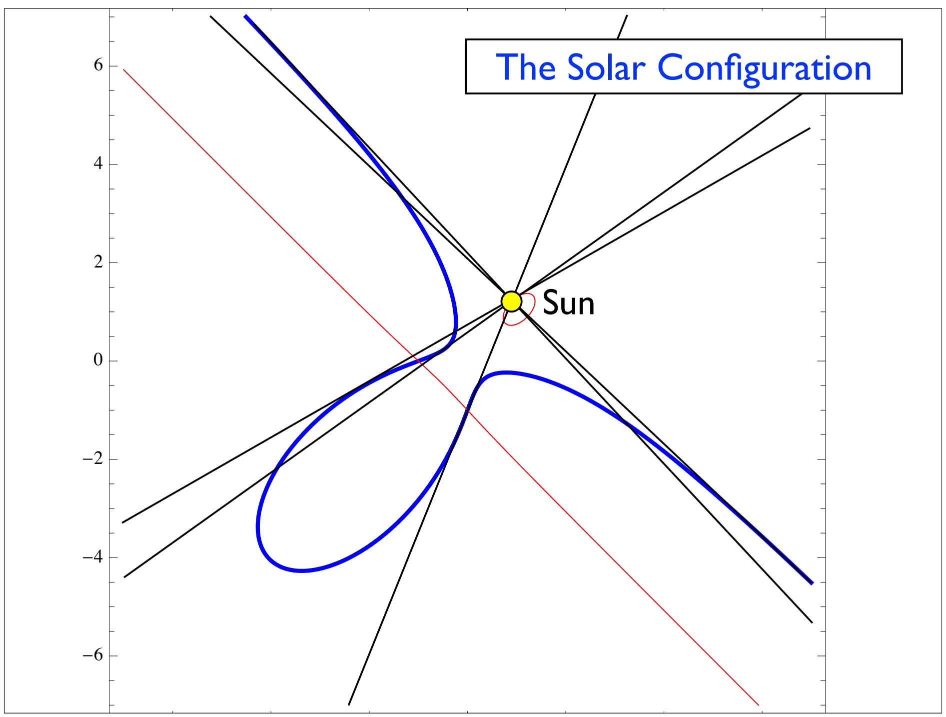




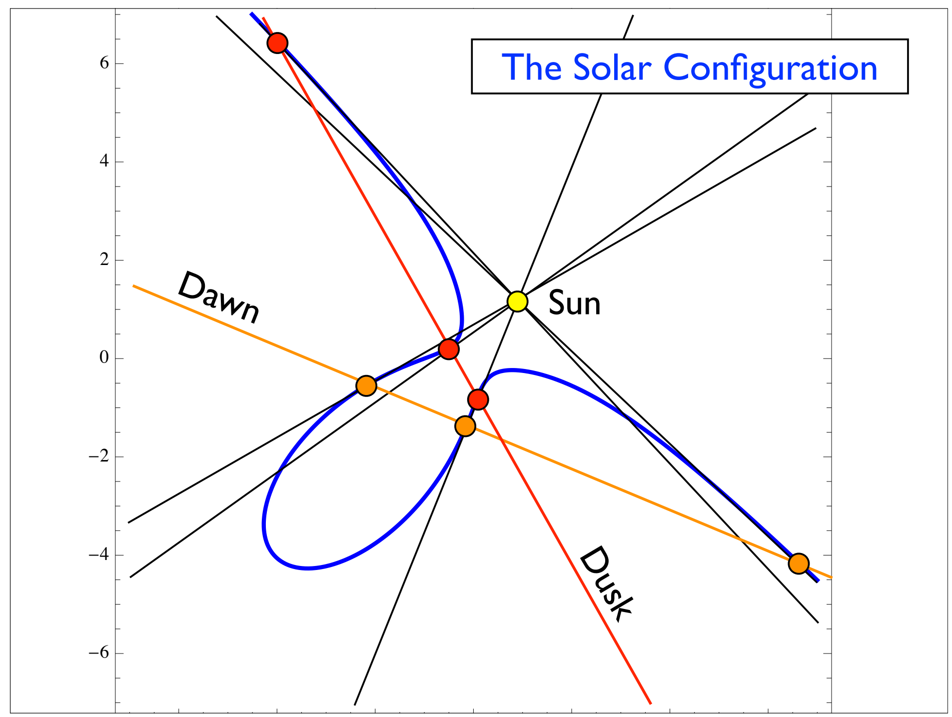
The Hessian
= Z(det D²f)
A
HA
HA = {S : Pol(A,S) is 2 lines}



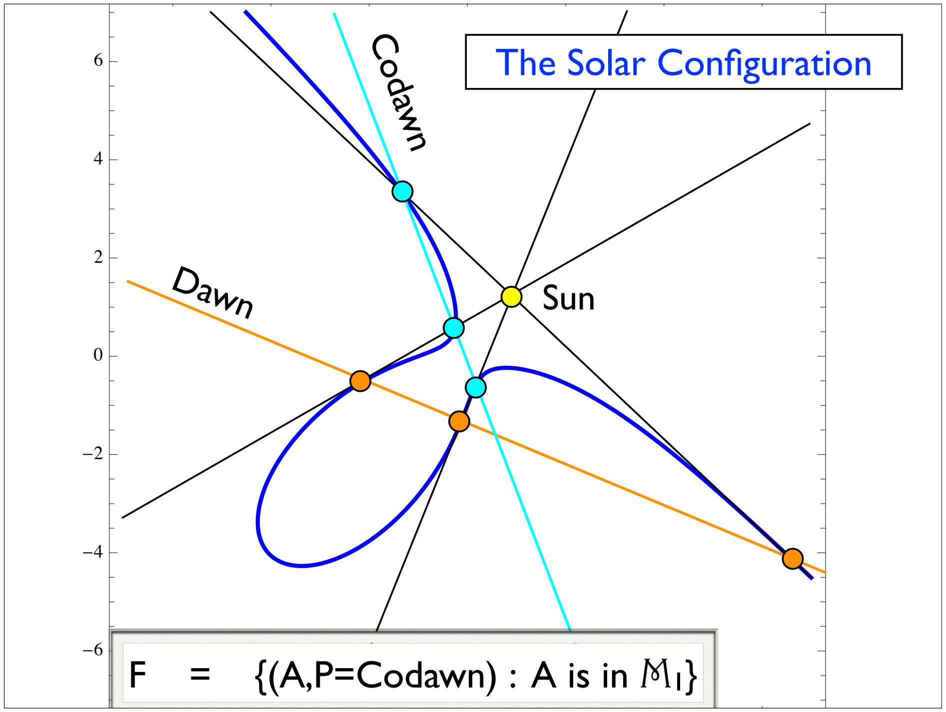
The Solar Configuration

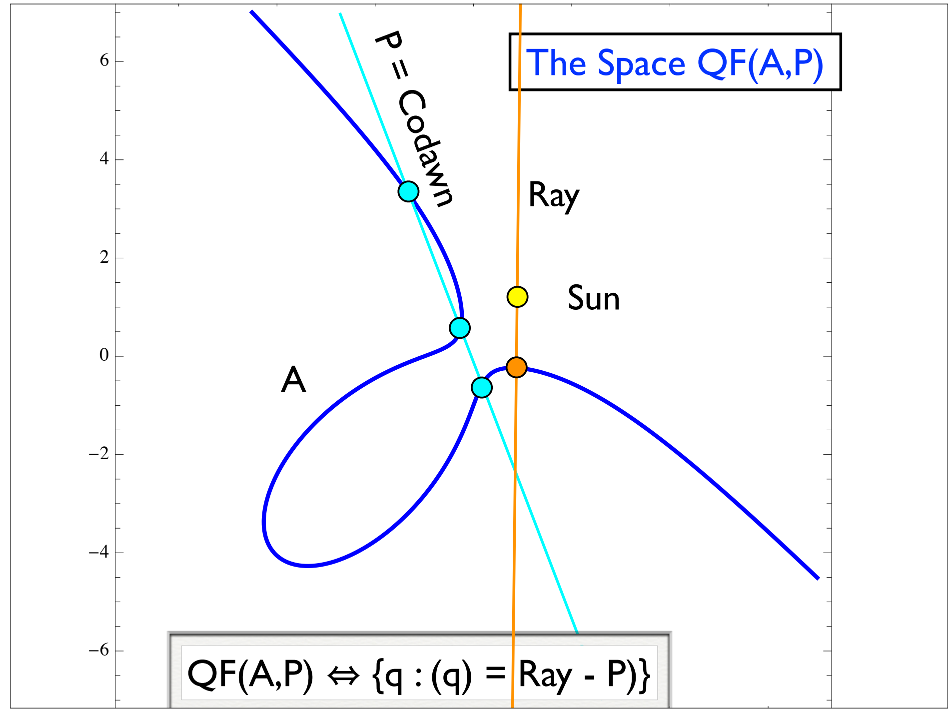


The Solar Configuration
Sun



The Solar Configuration
Sun
Dawn
Dusk





The gothic locus $\Omega G \subset \Omega\mathcal{M}_4$

Definition: $\Omega G = \sqrt{QF}$

(X, ω) in ΩG $\Leftrightarrow$ $(A,q) = (X/J, \omega^2/J)$ in QF

Theorem:

ΩG is an $SL_2(\mathbb{R})$ invariant subvariety of $\Omega\mathcal{M}_4$ of dimension 4

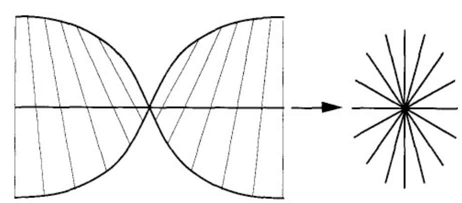
Cor:

G is geodesically ruled

From G back to F

(X, ω) in ΩG → $(A,q) = (X/J, \omega^2/J)$ in QF

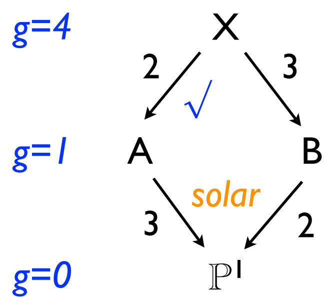
G
ruled



F

Corollary: F is totally geodesic

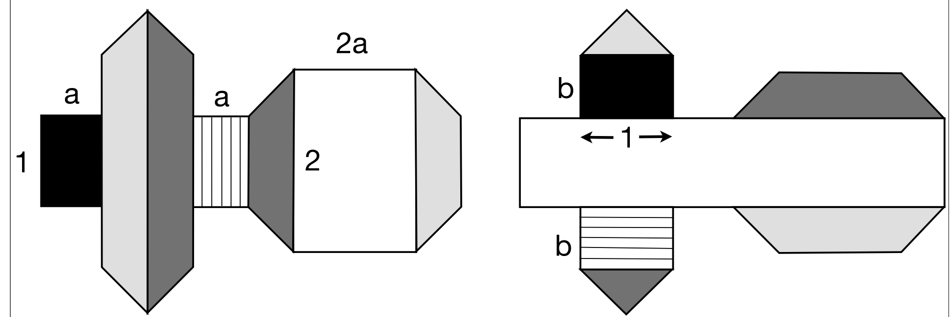
Proof of $SL_2(\mathbb{R})$ invariance of ΩG:



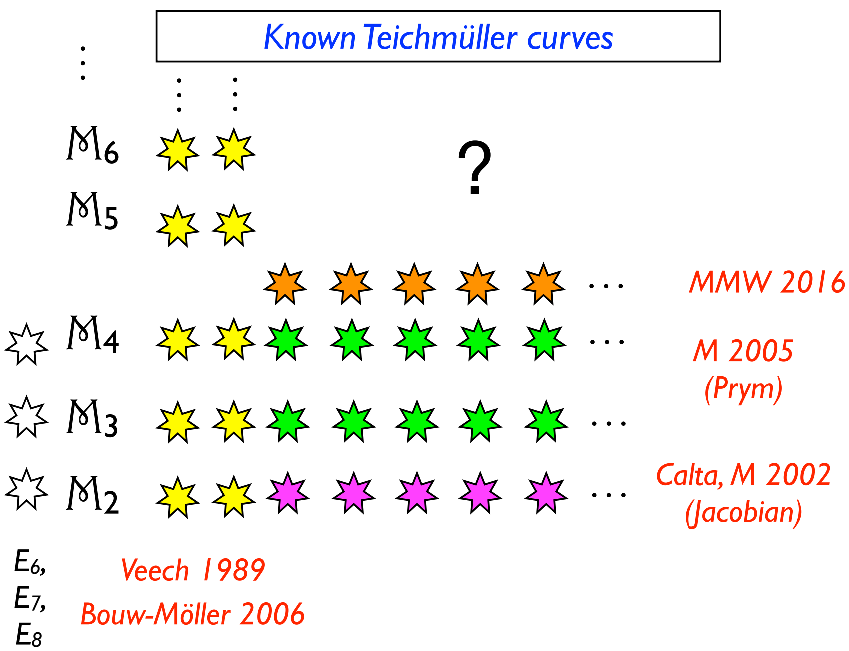
$$\begin{array}{ccccc} & & & & \omega \\ Jac(X) & \sim & A & \times B \times & C \\ 4 & = & 1 & + 1 + & 2 \end{array}$$

- *Variety ΩG is linear and defined/$\mathbb{R}$ in period coordinates on $\Omega\mathcal{M}_4$!*

Cathedral polygons

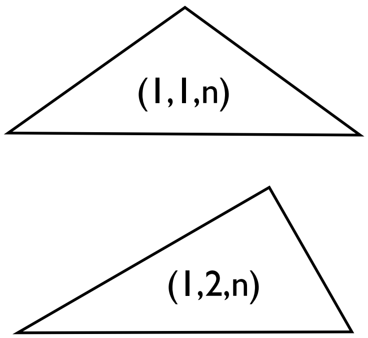


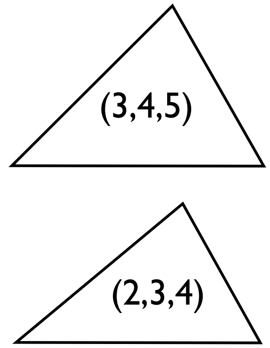
Complement.
We have a new infinite series of Teichmüller curves $G_D \subset G \subset \mathcal{M}_4$.

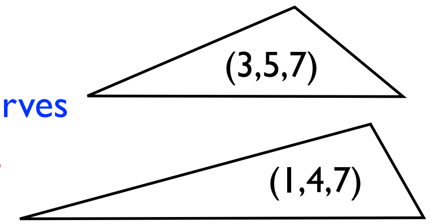


Does the flex surface fit into a broader pattern?

Classical perspective: Triangles



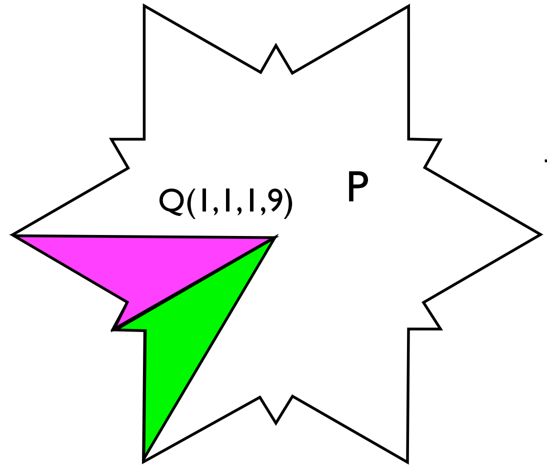




These generate Teichmüller curves

1989 Veech, Ward, Vorobets, Kenyon-Smillie, Hooper 2013

New perspective: Quadrilaterals



→ $(X,\omega) = (P, dz)/\sim$

in $\Omega\mathcal{M}_4$

$X: \quad y^6 = x(x-1)(x-t)$

$\omega: \quad C\, dx/y^5$

New perspectives on ΩG:

- The gothic locus is the *closure* of the $SL_2(\mathbb{R})$ orbits of the (1,1,1,9) quadrilateral forms.

($\mathbb{Z}/6$ symmetry)

- A Zariski open subset of ΩG coincides with an explicit space of dihedral forms.

(D_{12} symmetry)

- The variety of dihedral forms is linear in period coordinates.

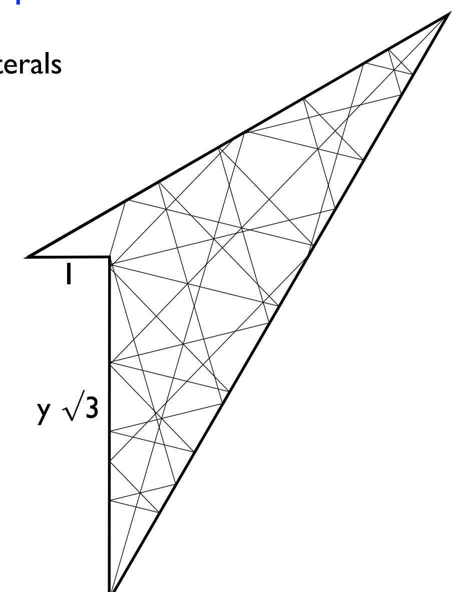
New optimal billiards

- Suitable (1,1,1,9) quadrilaterals have optimal dynamics and generate Teichmüller curves in $\mathcal{M}_4$.

$y^2 + (3c+1)y + c = 0$

some c in $\mathbb{Q}$

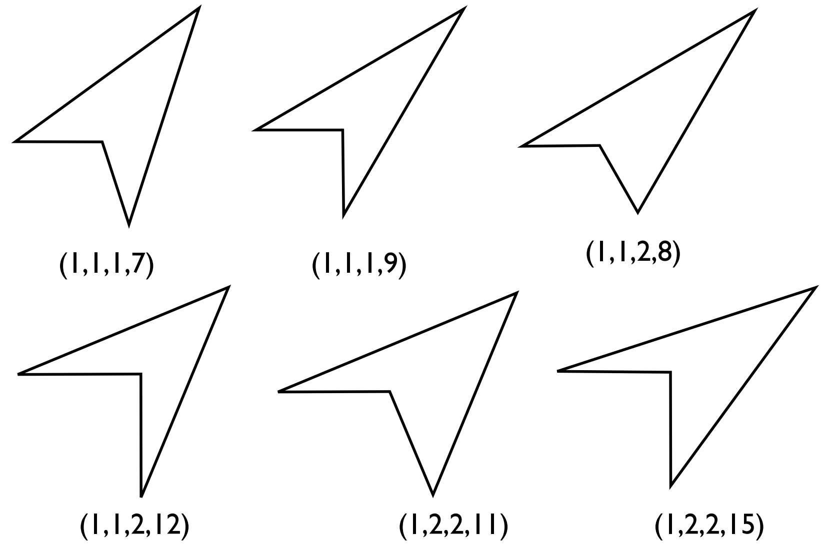
Shown: $y = 1 + \sqrt{2}$



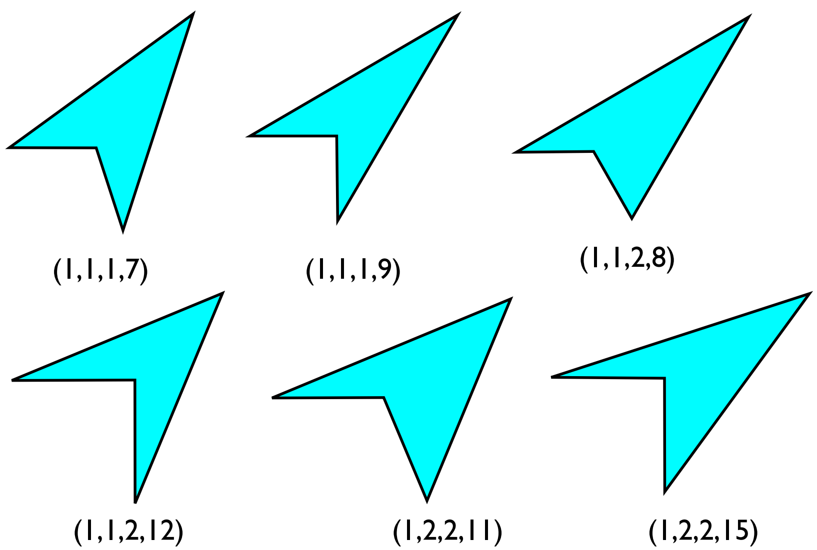
Does the (1,1,1,9) quadrilateral

fit into a broader pattern?

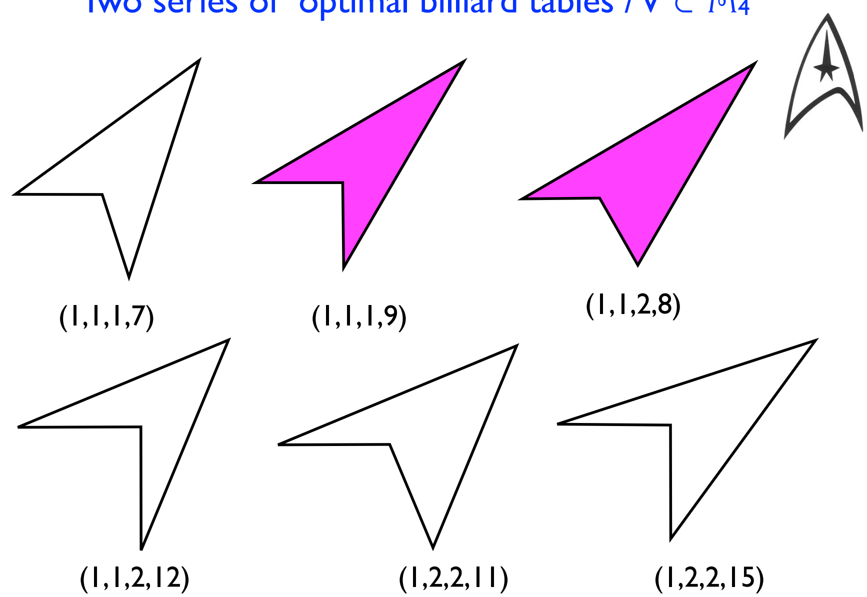
A suite of quadrilaterals



Six $SL_2(\mathbb{R})$ invariant 4-folds $\Omega G \subset \Omega\mathcal{M}_g$

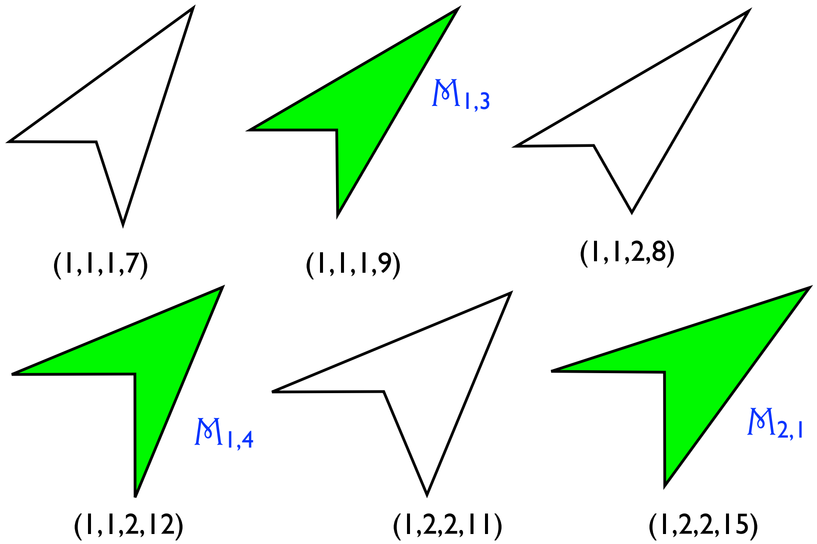


Two series of optimal billiard tables / $V \subset \mathcal{M}_4$



(cf. Bouw-Möller)

Three totally geodesic surfaces



Pentagons, *reprise*

- The closure of the $SL_2(\mathbb{R})$ orbits of the (1,1,2,2,12) pentagonal forms gives a new, 6-dimensional invariant subvariety of $\Omega\mathcal{M}_4$.