Billiards and curves in moduli space

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Avila, Hubert, Kenyon, Kontsevich, Lanneau,
Masur, Smillie, Yoccoz, Zorich, ...

A Little History

$$\int \frac{d\!\!\!/\,dx}{Q(x)^{1/d}}$$

$Q(x)$ a polynomial

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$$

$$\int \frac{dx}{\sqrt[3]{1-x^3}} =$$

$$\frac{3\sqrt[3]{\frac{x-1}{1+\sqrt[3]{-1}}+1}\sqrt[3]{\frac{x-1}{1-(-1)^{2/3}}+1}(x-1)F_1\left(\frac{2}{3};\frac{1}{3},\frac{1}{3};\frac{5}{3};-\frac{x-1}{1-(-1)^{2/3}},-\frac{x-1}{1+\sqrt[3]{-1}}\right)}{2\sqrt[3]{1-x^3}}$$

F_1 = Appel hypergeometric function

A Little History

$$\int \frac{dx}{Q(x)^{1/d}}$$
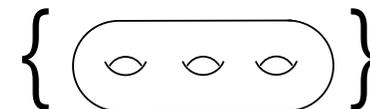
$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$$

(X,ω) = (The curve $y^d = P(x)$, the form dx/y)

periods of (X,ω) = $\left\{ \int_C \omega \right\}$

→ *Riemann surfaces, homology, Hodge theory,*
automorphic forms, ...

Moduli space

$\mathcal{M}_g$ = moduli space of Riemann surfaces X of genus g

$$\left\{ \;\bigcirc\!\!\!\diamond\;\diamond\;\diamond\; \right\}$$

-- a complex variety, dimension 3g-3

Teichmüller metric: every holomorphic map

$$f : \mathbb{H}^2 \to \mathcal{M}_g$$

is distance-decreasing.

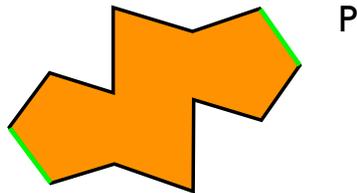*TOTALLY unsymmetric*

How to describe X in $\mathbb{M}_g$?

g=1: X = $\mathbb{C}/\Lambda$

g>1: X = ?   *Uniformization Theorem*

Every X in $\mathbb{M}_g$ can be built from a polygon in $\mathbb{C}$

P

X = P / gluing
   by translations

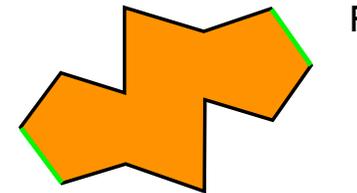## How to describe X in $\mathbb{M}_g$?

g=1: X = $\mathbb{C}/\Lambda$

g>1: X = ?   *Uniformization Theorem*

Every (X,ω) in $\Omega\mathbb{M}_g$ can be built from a polygon
$$\downarrow \quad \text{in } \mathbb{C}$$
$\mathbb{M}_g$

P

(X,ω) = (P,dz) / gluing
   by translations

Moduli space $\Omega\mathbb{M}_g$

> *Dynamical:*
> $SL_2(\mathbb{R})$ acts on $\Omega\mathbb{M}_g$

Polygon for A · (X,ω) = A · (Polygon for (X,ω))

Complex geodesics $f : \mathbb{H} \longrightarrow \mathbb{M}_g$

τ₁ — τ_1

τ₂ — τ_2

f(τ₁) — $f(\tau_1)$

f(τ₂) — $f(\tau_2)$
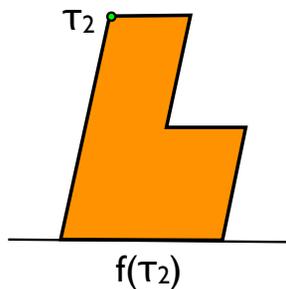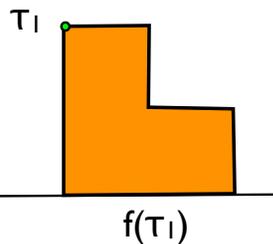
Teichmüller curves

$SL_2(\mathbb{R})$ orbit of (X,ω) in $\Omega\mathbb{M}_g$ projects to a

complex geodesic in $\mathbb{M}_g$:

$$\mathbb{H} \longrightarrow \mathbb{M}_g$$
$$\searrow \quad \nearrow f$$
$$V = \mathbb{H} \, / \, SL(X,\omega) \qquad \text{*stabilizer of*}$$
$$(X,\omega)$$

SL(X,ω) lattice ⇔ f : V → $\mathbb{M}_g$ is an algebraic,

isometrically immersed *Teichmüller curve*.

Rigidity Conjecture

The closure of any complex geodesic
$$f(\mathbb{H}) \subset \mathcal{M}_g$$
is an algebraic subvariety.

Celebrated theorem of Ratner (1995) $\Rightarrow$

true for $\mathbb{H} \longrightarrow$ *locally symmetric spaces*

$$X = K\backslash G/\Gamma$$

Complex geodesics in genus two

Theorem

Let $f : \mathbb{H} \to \mathcal{M}_2$ be a complex geodesic.
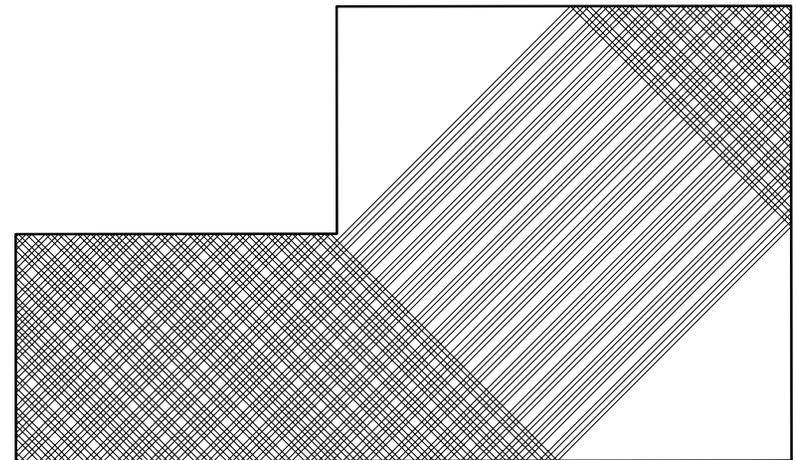Then $f(\mathbb{H})$ is either:

	dim
• A Teichmüller curve,	1
• A Hilbert modular surface H_D, or	2
• The whole space $\mathcal{M}_2$.	3

*Recent progress towards general g
Eskin – Mirzakhani*

Classification Problem

What *are* the Teichmüller curves $V \to \mathcal{M}_2$?
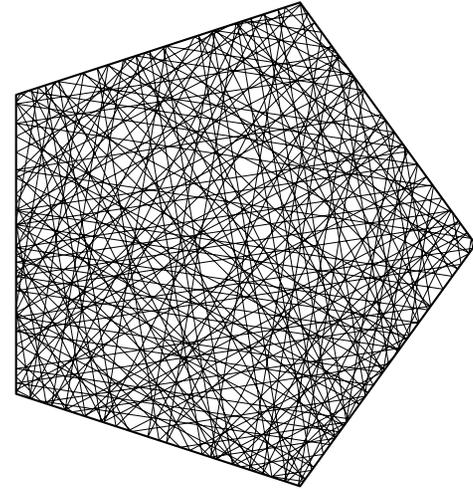

Billiards in polygons

Neither periodic nor evenly distributed
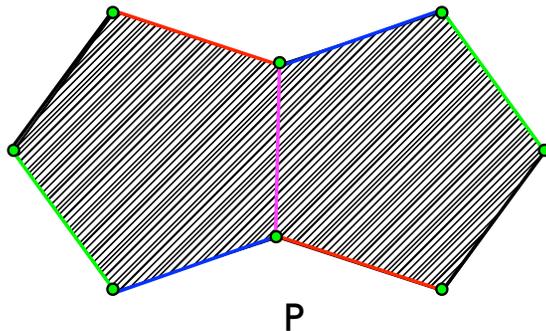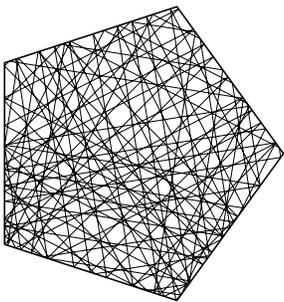
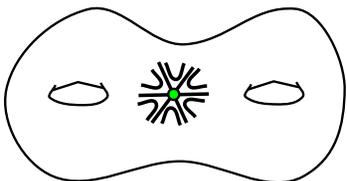## Billiard theorists

Optimal Billiards

(Veech)

Theorem. In a regular n-gon, every billiard path is either periodic or uniformly distributed.

Billiards and Riemann surfaces

P

$(X,\omega) = P/\sim$

X has genus 2
ω has just one zero!

P is a *Lattice Polygon*

$\Leftrightarrow$ SL(X,ω) is a lattice

$\Leftrightarrow (X,\omega)$ generates a Teichmüller curve
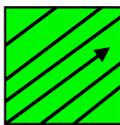
Theorem (Veech, Masur):

If P is a lattice polygon, then billiards in P is optimal.

(renormalization)

Optimal Billiards

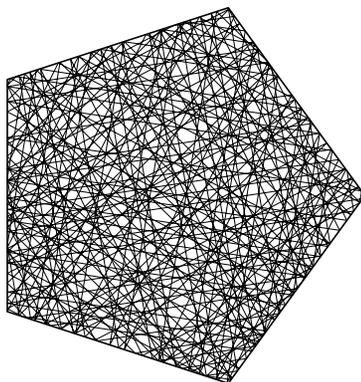Example: if $X = \mathbb{C}/\Lambda$, $\omega = dz$, then

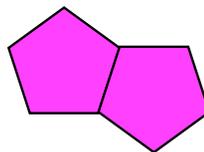$$SL(X,\omega) \simeq SL_2(\mathbb{Z})$$

Theorem (Veech, 1989): *For $(X,\omega) = (y^2 = x^n-1, dx/y)$, $SL(X,\omega)$ is a lattice.*
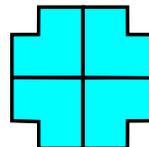
Corollary
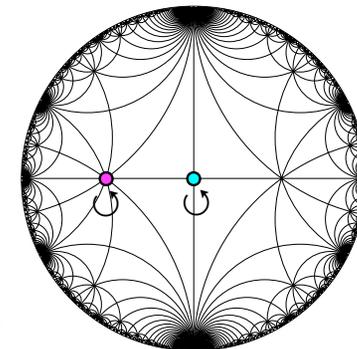*Any regular polygon is a lattice polygon.*

Explicit package: Pentagon example

$(X,\omega) = (y^2 = x^5-1, dx/y)$

$g \cdot (X,\omega)$

$\mathcal{M}_g$

$V = \mathbb{H}/SL(X,\omega) \subset SL_2(\sqrt{5})$

$= \langle a, b \rangle$

⇒ Direct proof that SL(X,ω) is a lattice

20th century lattice billiards
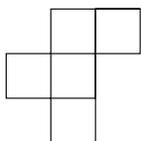
	Square	SL_2(Z)
	Tiled by squares	~ SL_2(Z)
	Regular polygons	~ (2,n, ∞) triangle group
	Various triangles	triangle groups
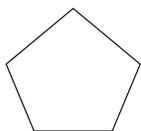
Genus 2

⤳ Regular 5- 8- and 10-gon

Problem

Are there infinitely many primitive Teichmüller curves V in the moduli space $\mathcal{M}_2$?

Jacobians with real multiplication

Theorem
(X,ω) generates a Teichmüller curve V⇒

Jac(X) admits real multiplication by $\mathcal{O}_D \subset \mathbb{Q}(\sqrt{D})$.

Corollary
V lies on a Hilbert modular surface

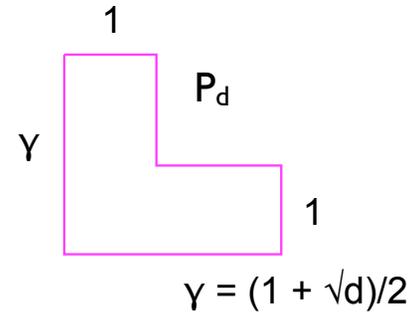$$V \subset H_D \subset \mathcal{M}_2$$

$$\shortparallel$$

$$\mathbb{H} \times \mathbb{H} \,/\mathrm{SL}_2(\mathcal{O}_D)$$

The Weierstrass curves

W_D = {X in $\mathbb{M}_2$: $\mathcal{O}_D$ acts on Jac(X) and its eigenform ω has a double zero.}

Theorem. *W_D is a finite union of Teichmüller curves.*

1

Y

P_d
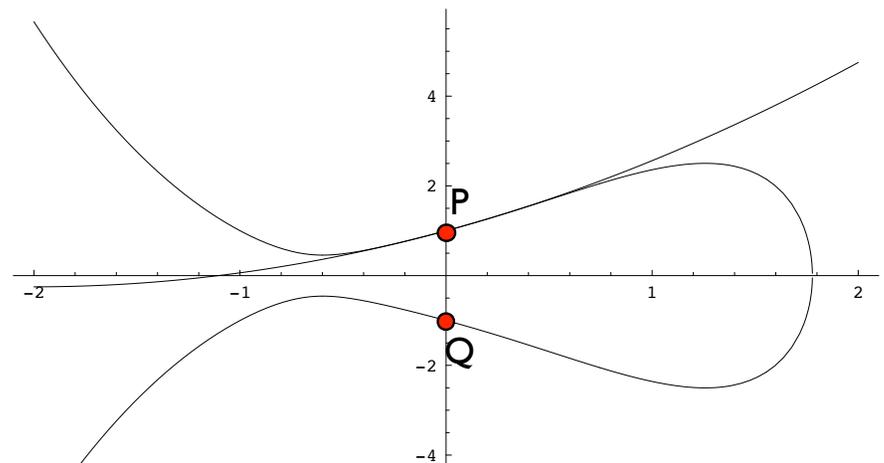
1

Y = (1 + √d)/2

Corollaries
- *P_d has optimal billiards for all integers d>0.*

- *There are infinitely many primitive V in genus 2.*

The regular decagon
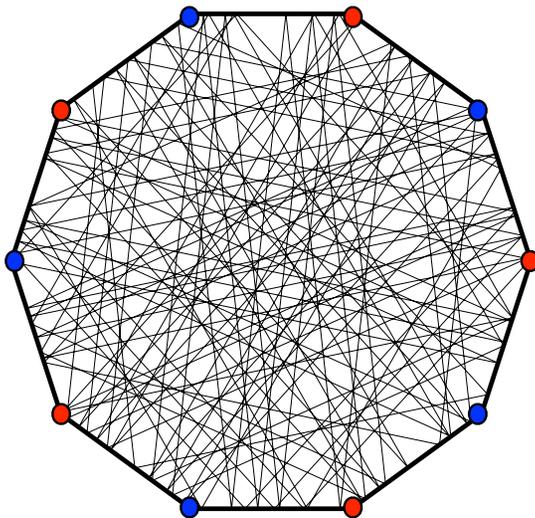
Theorem. *The only other primitive Teichmüller curve in genus two is generated by the regular decagon.*

Torsion divisors in genus two

P

Q

Theorem (Möller) *(X,ω) generates a Teichmüller curve ⇒ [P-Q] is torsion in Jac(X)*

Teichmüller curves in genus 2

Theorem

The Weierstrass curves W_D account for all the primitive Teichmüller curves in genus 2 --

-- except for the curve coming from the regular decagon.
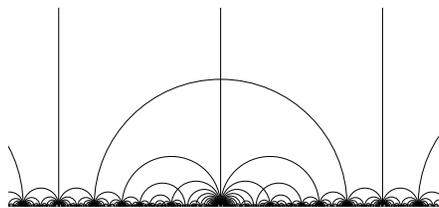
Mysteries

- Is W_D irreducible?
- What is its Euler characteristic?
- What is its genus?
- Algebraic points (X, ω) in W_D?
- What is $\Gamma = SL(X, \omega)$?

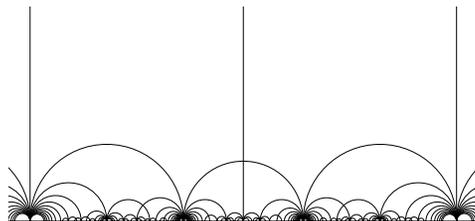$$W_D = \mathbb{H}/\Gamma, \quad \Gamma \subset SL_2(\mathcal{O}_D)$$

Classification Theorem   *M, 2004*

*W_D is connected
except when D=1 mod 8, D>9.*

(spin)

W_{17}^0

W_{17}^1

Euler characteristic of W_D

Theorem (Bainbridge, 2006)

$$\chi(W_D) = -\frac{9}{2}\chi(SL_2(\mathcal{O}_D))$$
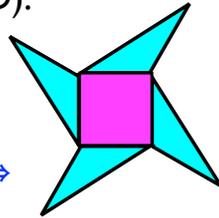
= coefficients of a modular form

Compare: $\chi(\mathbb{M}_{g,1}) = \zeta(1\text{-}2g)$ (Harer-Zagier)

*Proof: Uses cusp form on Hilbert modular surface with
$(\alpha) = W_D - P_D,$ where P_D is a Shimura curve*

Elliptic points on W_D

Theorem (Mukamel, 2011)

The number of orbifold points on W_D is given by a sum of class numbers for $\mathbb{Q}(\sqrt{-D})$.

Proof: (X,ω) correponds to an orbifold point $\Rightarrow$

X covers a CM elliptic curve $E \Rightarrow$

(X,ω), $p: X \to E$ and $Jac(X)$ can be described explicitly.

Genus of W_D

D	$g(W_D)$	$e_2(W_D)$	$C(W_D)$	$\chi(W_D)$	D	$g(W_D)$	$e_2(W_D)$	$C(W_D)$	$\chi(W_D)$
5	0	1	1	$-\frac{3}{10}$	52	1	0	15	-15
8	0	0	2	$-\frac{3}{4}$	53	2	3	7	$-\frac{21}{2}$
9	0	1	2	$-\frac{1}{2}$	56	3	2	10	-15
12	0	1	3	$-\frac{3}{2}$	57	$\{1,1\}$	$\{1,1\}$	$\{10,10\}$	$\{-\frac{21}{2},-\frac{21}{2}\}$
13	0	1	3	$-\frac{3}{2}$	60	3	4	12	-18
16	0	1	3	$-\frac{3}{2}$	61	2	3	13	$-\frac{33}{2}$
17	$\{0,0\}$	$\{1,1\}$	$\{3,3\}$	$\{-\frac{3}{2},-\frac{3}{2}\}$	64	1	2	17	-18
20	0	0	5	-3	65	$\{1,1\}$	$\{2,2\}$	$\{11,11\}$	$\{-12,-12\}$
21	0	2	4	-3	68	3	0	14	-18
24	0	1	6	$-\frac{9}{2}$	69	4	4	10	-18
25	$\{0,0\}$	$\{0,1\}$	$\{5,3\}$	$\{-3,-\frac{3}{2}\}$	72	4	1	16	$-\frac{45}{2}$
28	0	2	7	-6	73	$\{1,1\}$	$\{1,1\}$	$\{16,16\}$	$\{-\frac{33}{2},-\frac{33}{2}\}$
29	0	3	5	$-\frac{9}{2}$	76	4	3	21	$-\frac{57}{2}$
32	0	2	7	-6	77	5	4	8	-18
33	$\{0,0\}$	$\{1,1\}$	$\{6,6\}$	$\{-\frac{9}{2},-\frac{9}{2}\}$	80	4	4	16	-24
36	0	0	8	-6	81	$\{2,0\}$	$\{0,3\}$	$\{16,14\}$	$\{-18,-\frac{27}{2}\}$
37	0	1	9	$-\frac{15}{2}$	84	7	0	18	-30
40	0	1	12	$-\frac{21}{2}$	85	6	2	16	-27
41	$\{0,0\}$	$\{2,2\}$	$\{7,7\}$	$\{-6,-6\}$	88	7	1	22	$-\frac{69}{2}$
44	1	3	9	$-\frac{21}{2}$	89	$\{3,3\}$	$\{3,3\}$	$\{14,14\}$	$\{-\frac{39}{2},-\frac{39}{2}\}$
45	1	2	8	-9	92	8	6	13	-30
48	1	2	11	-12	93	8	2	12	-27
49	$\{0,0\}$	$\{2,0\}$	$\{10,8\}$	$\{-9,-6\}$	96	8	4	20	-36

Corollary

W_D has genus 0 only for D<50

(table by Mukamel)

Algebraic points on W_D

$$X \in \mathbb{M}_2$$

D=5	$y^2 = x^5 - 1$
D=8	$y^2 = x^8 - 1$
D=13	$y^2 = (x^2-1)(x^4 - ax^2 + 1)$
	$a = 2594 + 720\sqrt{13}$
	....
D=108	$96001 + 48003\, a + 3\, a^2 + a^3 = 0$

Mukamel

Computing W_D
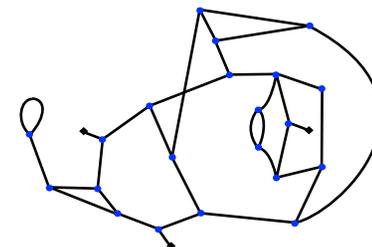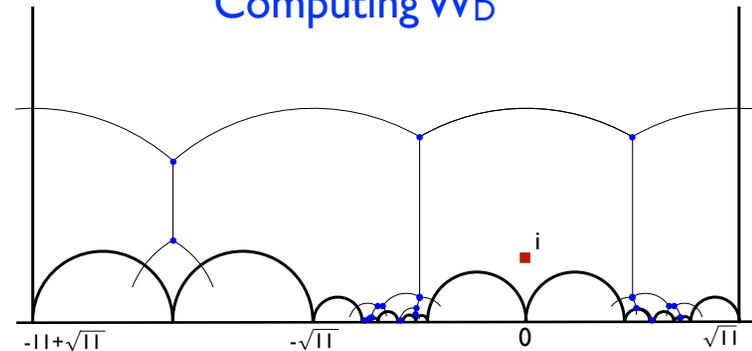
Mukamel

D=44

Higher genus?

Conjecture:

There are only finitely many Teichmüller curves in $\mathbb{M}_g$

with deg(trace field SL(X,ω)) = g = 3 or more.

(avoids echos of lower genera)

[Rules out quadratic fields]

Theorem (Möller, Bainbridge-Möller)

Finiteness holds...
  for hyperelliptic statum (g-1,g-1)
  for g=3, stratum (3,1)

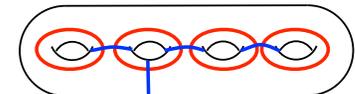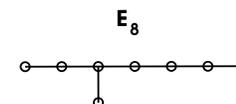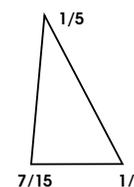**Methods:** Variation of Hodge structure; rigidity theorems of Deligne and Schmid; Neron models; arithmetic geometry

- Jac(X) admits real multiplication by K,
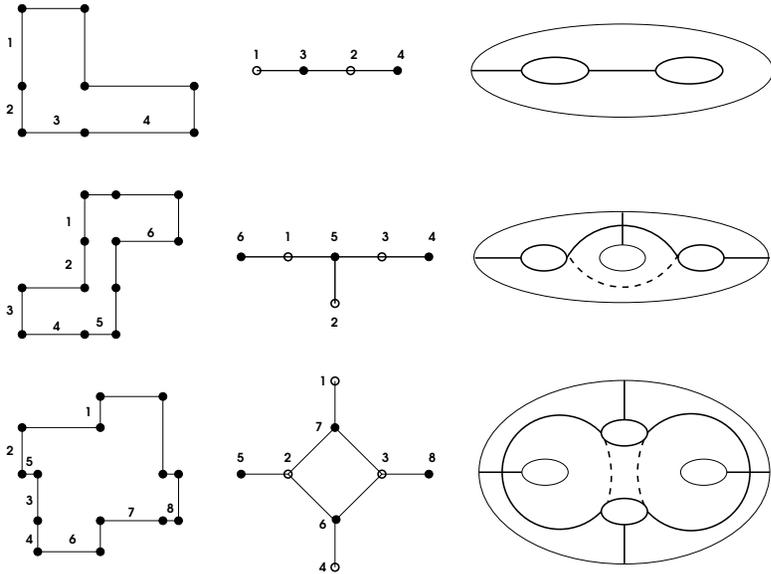- P-Q is torsion in Jac(X) for any two zeros of ω.

However...

Theorem

There exist infinitely many primitive Teichmüller curves in $\mathbb{M}_g$ for genus g = 2, 3 and 4.

Exceptional triangular billiards
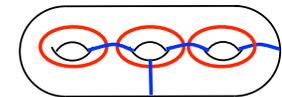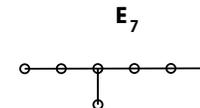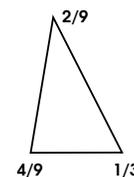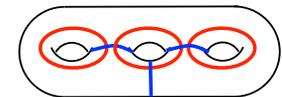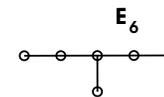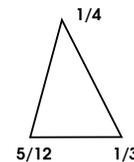
Prym systems in genus 2, 3 and 4

W_D for g=3,4: Lanneau--Nguyen   *but still quadratic fields*

Higher genus?

Question.

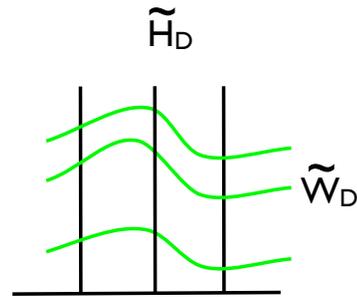*Are there only finitely many primitive Teichmüller curves*

in $\mathbb{M}_g$ for each g ≥5?

What about the Hilbert modular surfaces
$H_D \subset \mathbb{M}_2$?

$\mathbb{H} \times \mathbb{H}$ *foliated by complex geodesics*

↓

$H_D \subset \mathbb{M}_2$

$\widetilde{H}_D$

$\widetilde{W}_D$

each leaf is the graph of
a holomorphic function
$F: \mathbb{H} \to \mathbb{H}$

Pentagon-to-star map

F
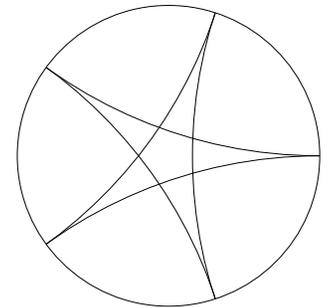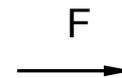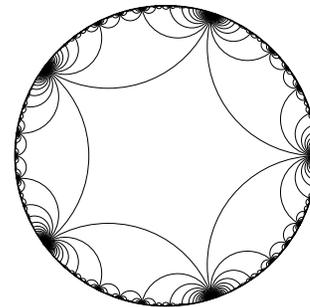
$\widetilde{W}_5$ = graph of F

Action on slices of H_D
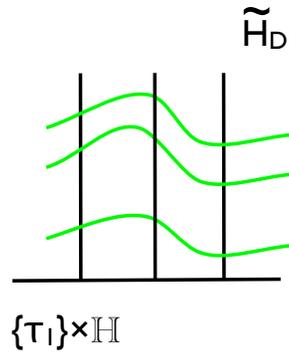
Slice $\{\tau_I\} \times \mathbb{H}$

$\rho = \int_a^b \omega$ = relative period
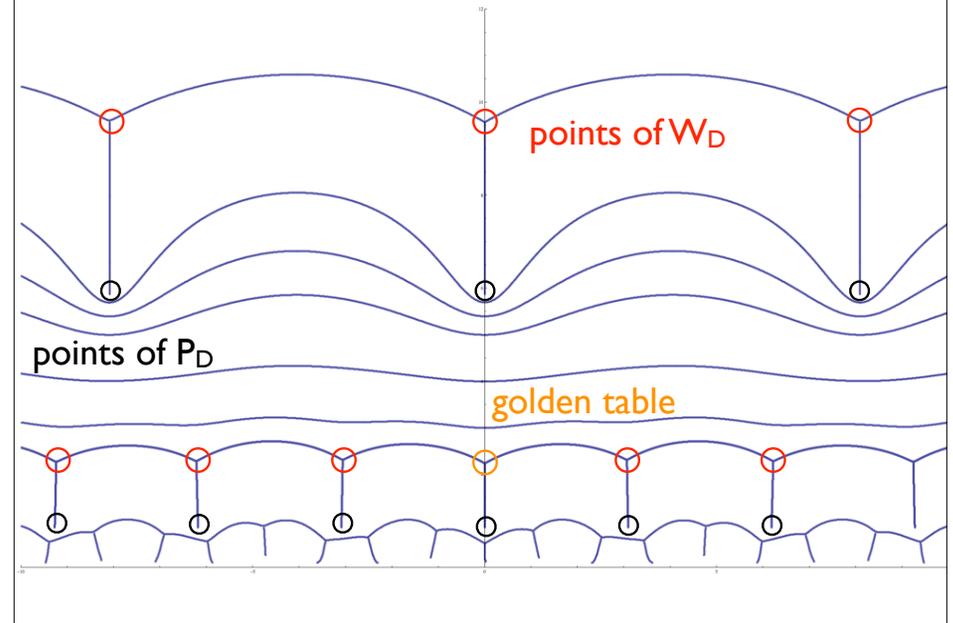
$q = (d\rho)^2$ quadratic differential

$SL(\mathbb{H}, q) = SL_2(\mathcal{O}_D)$

acts on slice
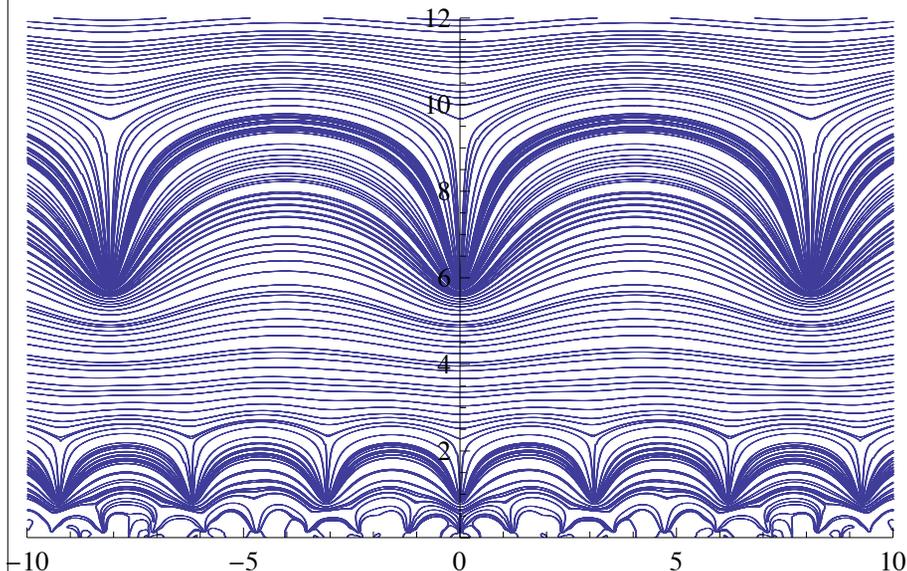
gives picture of action of $SL_2(R)$ on ΩM_2

$\widetilde{H}_D$

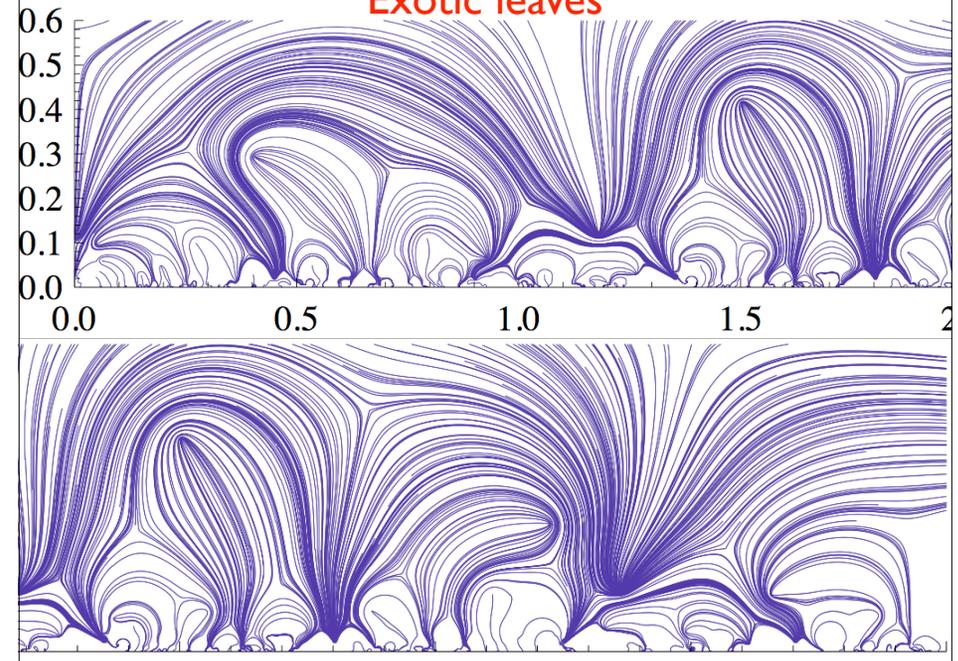$\{\tau_I\} \times \mathbb{H}$

Slice of H_D

points of W_D

points of P_D

golden table

Slice of Hilbert modular surface

D=5

$q = Q \mid \{\tau_1\} \times \mathbb{H}$
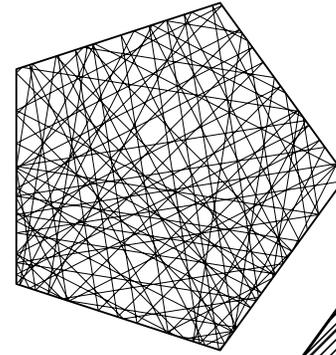
Exotic leaves

Billiards in a pentagon, reprise

Möller-Zagier formula

$$Q = \left(\prod_{m \text{ odd}} \frac{d\vartheta_m}{dz_2}(\tau, 0) \Big/ \prod_{m \text{ even}} \vartheta_m(\tau, 0) \right) d\tau_1^{-1} \, d\tau_2^2.$$

products taken over spin strs m
(6 odd, 10 even)

(Q) = W_D - P_D on the Hilbert modular
surface $X_D = \mathbb{H} \times \mathbb{H} \, / SL_2(\mathcal{O}_D)$