Billiards and the arithmetic of non-arithmetic groups

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Hecke, Rosen, Leutbecher, Cohen-Wolfart, Davis-Lelievre, ...

The Hecke groups G_q

$G_q = \langle S,T \rangle$, where

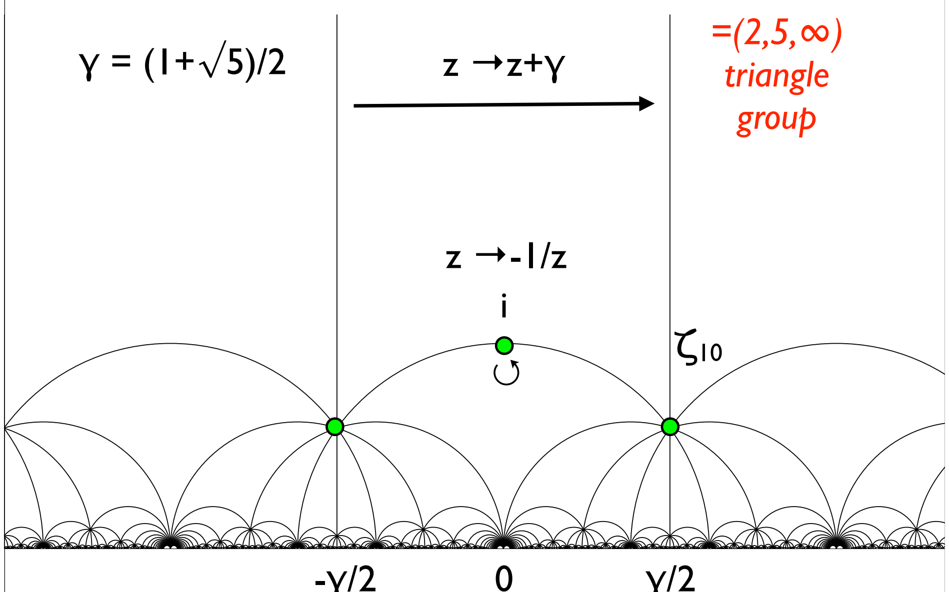
$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 2\cos \pi/q \\ 0 & 1 \end{pmatrix}.$$

Abgesehen von den vier Fällen $q = 3, 4, 6$ und ∞, ... ist so etwas wie ein Koeffizientengesetz der Matrizen zu G_q noch nicht bekannt. [2]

—Leutbecher, 1974.

[2] *Aside from the four cases $q = 3, 4, 6$ and ∞... no rule for the coefficients of the matrices in G_q is yet known.*

The golden Hecke group G_5



Matrix entries in G_5

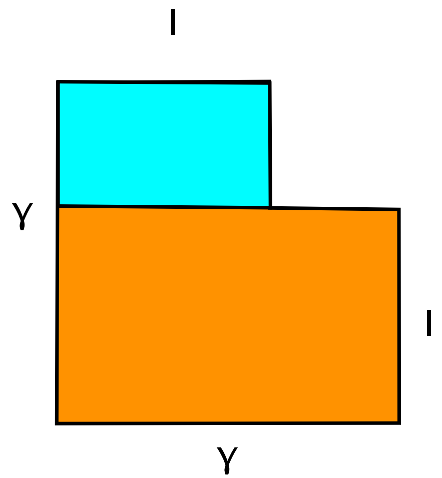
Theorem
Let $x = A + B\gamma$ be a matrix entry in G_5. Then $AB \geq 0$.

Cor.
$|x'| \leq |x|$.

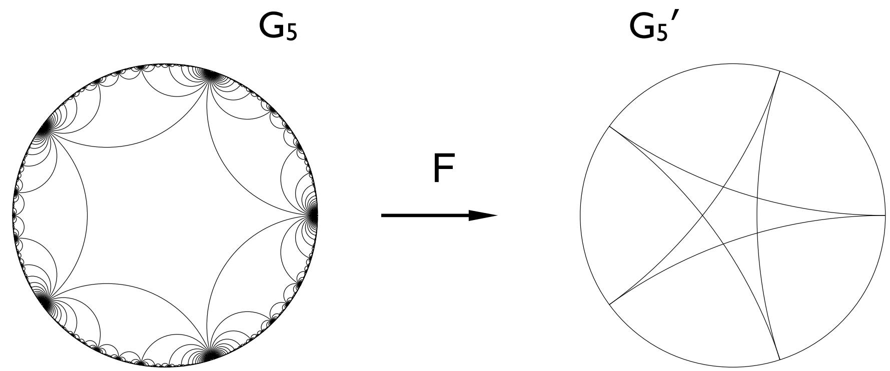
Complement.
For any g in G_5, we have $|\mathrm{Tr}\, g'| \leq |\mathrm{Tr}\, g|$ provided the latter is > 2.

Proofs:

$G_5 = SL(X,\omega)$. Use the golden table!



Holomorphic pentagon-to-star map

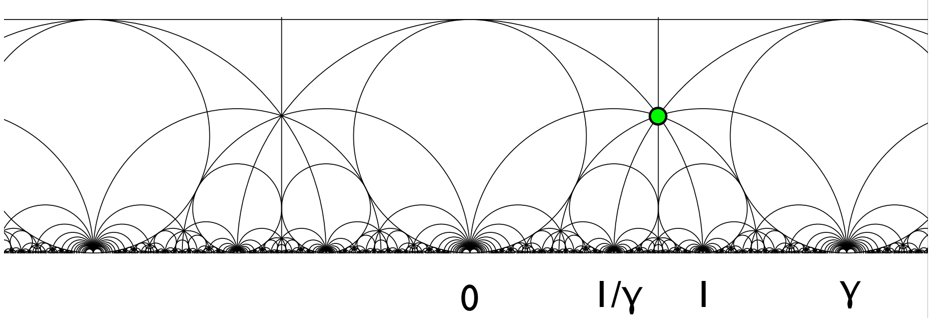


Cusps of the golden Hecke group

Theorem

The cusps of G_5 coincide with $K = \mathbb{Q}(\sqrt{5})$.

Leutbecher, M



Quadratic trace field

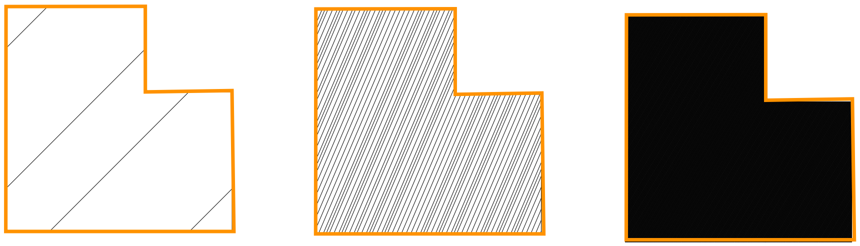
Consider a form (X,ω) of any genus g.

Theorem

If $SL(X,\omega)$ is a lattice with quadratic trace field K, then the cross-ratios of its cusps coincide with $K \cup \{\infty\}$.

Example: g=3, the regular 12-gon; handles G_{12}.

Simple slopes, long trajectories



s=1 — 4 segs

s=7/3 — 80 segs

s=5/3 — 1152 segs

s=89/55 — 10^{13} segs

Effective methods

Theorem

Let s = (a+bγ)/(c+dγ) with |a|,|b|,|c|,|d| < M.

Then the trajectory at slope s
in the golden L has length at most
$\exp(c(\log M)^2)$.

Sharp for s a Fibonacci number.

The arithmetic of G_5

Let $R = \{a'/a : a \neq 0$ is a matrix entry in $G_5\}$.

Theorem

I. R is contained in [-1,1].

II. The closure S of R is countable.

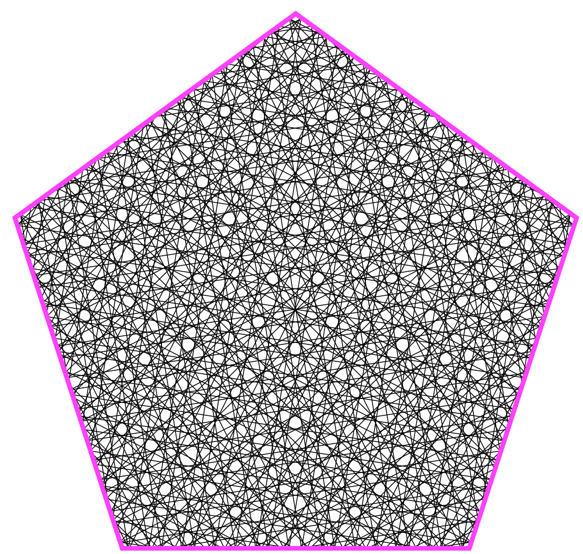
III. S is homeomorphic to $\omega^\omega+1$.

IV. S is equal to the semigroup generated by R.

Cor

There is no simple characterization of the
matrix coefficients of G_5.

Long periodic trajectories are uniformly distributed...aren't they?



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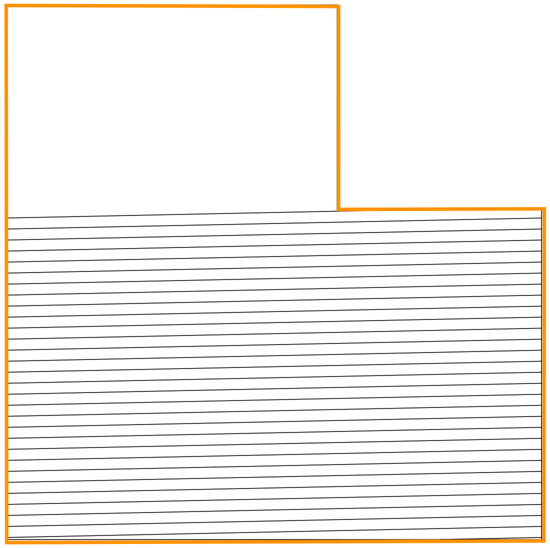
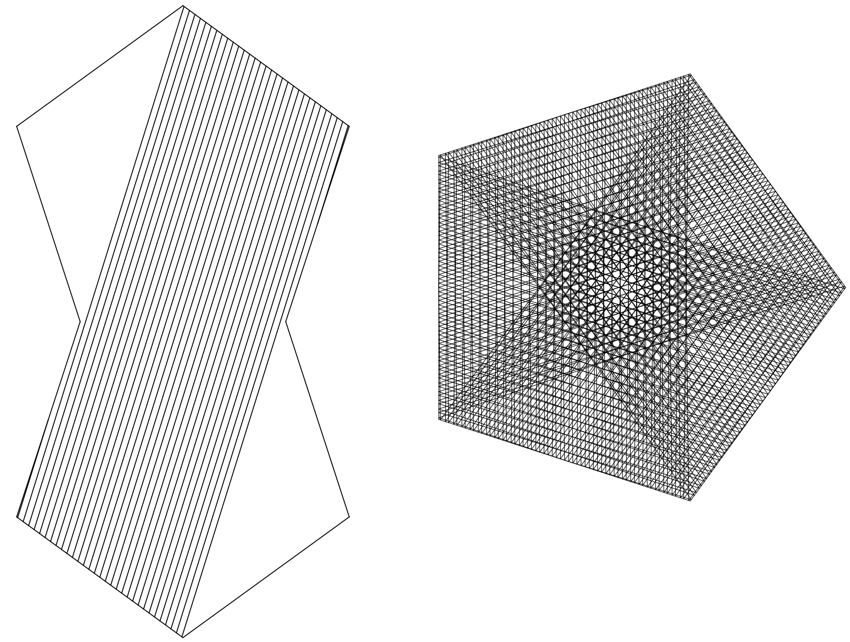
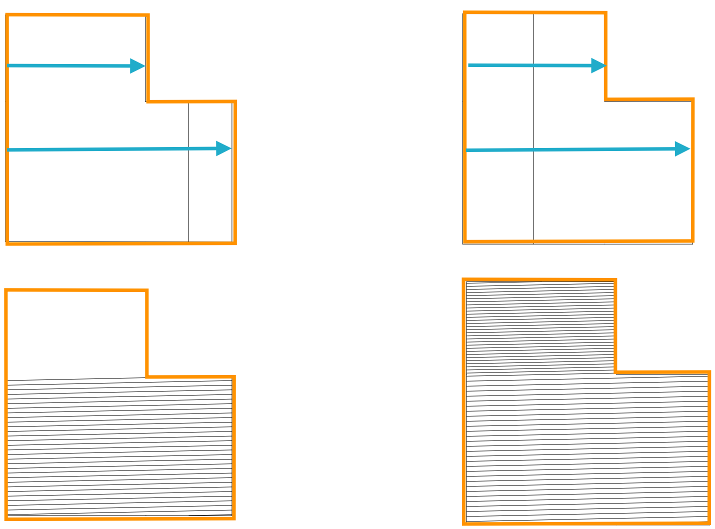




Twists



Non-optimality of optimal billiards

Q. of Davis-Lelievre

Let $m(s)$ = probability measure on a periodic trajectory of slope s in $\mathbb{Q}(\sqrt{5})$.

Let $M = \{\text{all } \lim m(s_n) \text{ for } s_n \to 0.\}$

Theorem (M)

M is homeomorphic to $\omega^{\omega}+1$.

Coxeter diagrams and Teichmüller curves

Every Teichmüller curve V can be specified by a curve system (A_i), (B_j).

$$\text{Gives } \langle D\tau_A, D\tau_B \rangle = \Gamma \subset SL(X, \omega).$$

Usually of infinite index!

Modular symbols organize all curves systems attached to V.

`New' proof of Veech dichotomy