Course Outline
Banach algebras and spectral theory
Math 206, Fall 1992, Berkeley CA
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Texts: Berberian, *Lectures on functional analysis and operator theory.* Riesz and Nagy, *Functional analysis.* Wermer, *Banach algebras and several complex variables.* Reed and Simon, *Methods of modern mathematical physics I: Functional analysis.* A. Connes, Geometrie non commutative. B. Weiss, Orbit equivalence of nonsingular actions, in *Théorie Ergodique*, L'Enseignement Mathématique 29 (1981), pp.77-107.

1. Examples of Banach spaces: $L^p(\mathbb{R})$, $\ell^p(\mathbb{N})$, $C(X)$, $C^k(\mathbb{R})$, $H^p(\Delta)$, $B(\mathcal{H})$.

2. Aperitif. Theorem (Wiener) Let $f(t) = \sum a_n \exp(int)$ be a continuous periodic function with $\sum |a_n| < \infty$. If $f(t) \neq 0$, then $1/f(t) = \sum b_n \exp(int)$ with $\sum |b_n| < \infty$.

3. Discussion: $A = \ell^1(\mathbb{Z})$ is a Banach algebra with respect to convolution, isomorphic to the algebra of periodic functions with absolutely summable power series. If $1/f$ is not in this algebra, then f generates an ideal fA which is contained in a maximal ideal M. One can show that A/M is isomorphic to the complex numbers and the map $A \to A/M$ is continuous. But the only such maps come from point evaluations, as can be seen by checking on $\exp(int)$ which generates a dense subalgebra. Thus $f(t) \neq 0$ implies f is contained in no such ideal and hence $1/f$ lies in A.

4. Topological groups. Given any neighborhood U of the identity, there exists a neighborhood V with $V \cdot V \subset U$. A topological group G is separated iff single points are closed sets. Quotient groups and uniform structures.

5. Theorem (Birkhoff-Kakutani) If G has a countable base at 1 and is separated, then G admits a left-invariant metric.

6. Example: $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$, the p-adic integers, form a compact abelian infinite totally disconnected group.

7. Topological vector spaces; Banach space duals and weak topologies. Quotient spaces.

8. Examples: $(s) = \mathbb{R}^{\mathbb{N}}$ is a complete metrizable TVS with $d(0, x_n) = \sum 2^{-n} |x_n|/(1 + |x_n|)$.

   The space (S) = measurable maps from $[0, 1]$ to $\mathbb{R}$ with $d(0, f) = \int_0^1 |f|/(1 + |f|)$. Convergence in (S) is equivalent to convergence in measure.

   Every neighborhood U of the origin in (s) or (S) contains a line. The functionals $\phi(x) = x_n$ span $(s)^*$. The dual space $(S)^*$ is trivial. Each space forms an algebra, and the group of invertible elements is not open.

9. Finite dimensional spaces: a separated locally compact TVS is finite dimensional.

10. Normed spaces. E is normable iff E is separated and there exists a bounded convex neighborhood of the origin.

11. The Hahn Banach Theorem: let A be an open nonempty convex set in a TVS E, and let M be a subspace disjoint from A. Then $M \subset H$ a closed hyperplane, also disjoint from E.

12. Traditional version: Given a closed subspace F of a Banach space E, and an element $\phi \in F^*$, there is an extension to an element $\psi \in E^*$ with $||\phi|| = ||\psi||$. Corollary: E^* separates points of E.

13. Example: there is a function $\phi : \ell^{\infty}(\mathbb{N}) \to \mathbb{R}$ that restricts to the usual limit on the convergent sequences.

14. Theorem (Alaoglu): The unit ball in the dual E^* of a Banach space is compact in the weak* topology.

15. Banach limits. There exists a translation invariant mean $m : \ell^{\infty}(\mathbb{Z}) \to \mathbb{R}$; that is m is a linear functional with $m(a_n) \geq 0$ if $a_n \geq 0$, $m(1) = 1$ and $m(a_{n+k}) = m(a_n)$. Proof: consider averaging over intervals $[-N, N]$, and pass to a subnet to find a limit as guaranteed by Alaoglu's theorem.

   More generally, any abelian group is amenable.

16. Locally convex TVS. Examples: (s) is locally convex, but (S) is not.

17. Theorem: A convex set in a locally convex separated TVS is closed iff it is weakly closed.

18. Theorem (Krein-Milman): Let A be a compact convex subset of a separated locally convex TVS. Then A is the closed convex hull of its extreme points. Choquet theory.

19. Banach spaces and Hilbert spaces. The map $X \to X^{**}$ is an isometry.

20. Adjoints on $\mathcal{L}(\mathcal{H})$. Basic facts: $||T^*|| = ||T||$, $||T||^2 = ||T^*T||$.

21. Baire category. Liouville numbers are transcendental. Diophantine numbers (of arbitrary exponent) have full measure but are meager.

22. Three basic principles: the uniform boundedness principle; the open mapping theorem; the closed graph theorem. (Open mapping implies closed graph by considering the graph itself as a Banach space, mapping injectively to the domain of the map.)

23. Theorem: $c_0 \subset \ell^\infty$ has no complementary subspace.

24. Theorem: every self-adjoint linear map $T : \mathcal{H} \to \mathcal{H}$ is bounded. Proof: It suffices to show the graph is closed. If $f_n \to f$ and $Tf_n \to g$, then for all h, $< Tf, h >=< f, Th >= \lim < f_n, Th >= \lim < Tf_n, h >=< g, h >$. Thus $Tf = g$.

25. Banach algebras A. Adjunction of identity. Right and left inverses; if xy and yx are both invertible, then so are x and y.

26. Theorem: $||x|| < 1$ implies $1 + x$ is invertible. Corollaries: the group of units is an open topological group in A.

27. Resolvent and spectrum. Theorem: The spectrum is not empty. Corollary (Gelfand-Mazur): A division ring A which is a Banach algebra over $\mathbb{C}$ is isomorphic to $\mathbb{C}$. Proof: otherwise, $\phi((\lambda - x)^{-1})$ would be a holomorphic function tending to zero at infinity for each $\phi \in A^*$.

28. Gelfand Representation Theorem: let A be a commutative Banach algebra with identity. Let $\mathcal{M}$ be its space of maximal ideals (equivalently, characters); this is a compact Hausdorff space in the weak* topology.

Then there is a natural algebra homomorphism $A \to \widehat{A} \subset C(\mathcal{M})$ such that for each a in A, the range of $\widehat{a}$ is the spectrum of a, $\widehat{A}$ is a full subalgebra, $\widehat{A}$ separates points of $\mathcal{M}$, and the kernel consists of the radical of A.

29. Rational functional calculus. There is a unique homomorphism $\mathbb{C}(t; \sigma(a)) \to A$, from the algebra of rational functions with poles outside the spectrum of a into A, sending 1 to 1 and t to a. The image is the smallest full subalgebra of A containing 1 and a.

    Spectral mapping theorem: $\sigma(f(a)) = f(\sigma(a))$.

30. Formula for the spectral radius: $r(x) = \lim ||x^n||^{1/n}$.

31. Example of nontrivial radical: the operator I on $C[0, 1]$ given by $(If)(x) = \int_0^x f(t)dt$ satisfies $||I^n||^{1/n} \to 0$.

32. Topological divisors of zero. Every x in the boundary of the group of units is a TDZ.

33. Spectrum and inclusion. If $B \subset A$ is a closed subalgebra with identity of a Banach algebra with identity, then for all x in B, $\sigma_B(x) \supset \sigma_A(x)$, and $\partial\sigma_B(x) \subset \partial\sigma_A(x)$.

34. Question: How to compute the operator norm of a matrix $A \in M_n(\mathbb{C})$?
    Answer. $||A||^2 = ||A^*A|| = \sup |\lambda|$ over eigenvalues of A^*A.

35. C^* algebras: basic facts. For any normal element (meaning x commutes with x^*, we have $||x|| = r(x)$. For any self-adjoint element, $\sigma(x) \subset \mathbb{R}$.

36. Examples of C^* algebras: $C[0, 1]$, $L^\infty[0, 1]$, $M_n(\mathbb{C})$, $\mathcal{L}(\mathcal{H})$.

37. Adjunction of the identity: every C^* algebra embeds isometrically in a C^* algebra with identity.

38. Theorem (Stone-Weierstrass): Let X be a compact Hausdorff space, A a closed subalgebra of the real-valued continuous functions on X which contains 1 and separates points. Then $A = C_{\mathbb{R}}(X)$.

    Proof (de Brange; cf. Wermer). Let A be such an algebra. If A is a proper closed subspace of $C(X)$, we can choose μ to be an extreme point of the measures of total variation one which vanish on all elements

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of A. Since $\mu(1) = 0$ the support of μ contains at least two points x and y. Since A separates points, we can find an f in A with $f(x) = 1$ and $f(y) = 0$. Then μ is a convex combination of $f\mu$ and $(1 - f)\mu$, suitably scaled. Since μ is an extreme point, we have f is constant a.e. with respect to μ, a contradiction.

39. Theorem (Gelfand-Naimark): Every commutative C^* algebra with identity is isometrically isomorphic to $C(\mathcal{M})$.

    The continuous functional calculus for any normal element is a corollary.

40. Theorem. Any map between C^* algebras sending identity to identity is continuous; in fact norm non-increasing.

41. The positive elements in a C^* algebra are those self-adjoint elements with spectra in the non-negative reals. Theorem. The positive elements are exactly those which can be written as $x = a^*a$.

    The positive elements form a convex cone.

42. Definition. A *state* ϕ on a C^* algebra is a continuous linear functional taking positive values on positive elements (more precisely $\phi(a^*a) \geq 0$). A state is *normalized* if $\phi(1) = 1$. Theorem. A linear functional ϕ is a state iff $\phi(1) = ||\phi||$.

    A normalized state on a commutative C^* algebra $A \cong C(X)$ is just a probability measure on X. Thus a state is a non-commutative measure.

43. Theorem. For all $a \geq 0$ in A, there exist a normalized state such that $\phi(a) = ||a||$. The proof is by the Hahn-Banach theorem, starting with a state on the commutative algebra generated by a.

44. The GNS (Gelfand-Naimark-Segal) construction: given a state ϕ, there is a naturally associated Hilbert space $\mathcal{H}_\phi$ and a norm-nonincreasing map $A \to \mathcal{L}(\mathcal{H}_\phi))$. The idea is to define an inner product by $<a, b> = \phi(b^*a)$.

45. Theorem: Every C^* algebra can be realized as a closed subalgebra of $\mathcal{L}(\mathcal{H})$ for some Hilbert space. The proof is to take the direct sum of $\mathcal{H}_\phi$ over all normalized states ϕ.

46. On $A = M_n(\mathbb{C})$, every positive element a determines a state by $\phi(b) = \mathrm{tr}(ab)$. All states arise in this way.

47. Let T be a self-adjoint operator in the C^* algebra $\mathcal{L}(H)$. Then: T is positive if and only if $< Tx, x >\geq 0$ for all x in H; and $||T|| = \sup_{||x||=1} | < Tx, x > |$.

48. The operator T is *compact* if the image of the unit ball under T has compact closure.

    Theorem: Given a compact self-adjoint operator T, there exists an orthonormal sequence of eigenvectors e_i and eigenvalues λ_i such that $Te_i = \lambda e_i$, $\lambda_i \to 0$ and $T|N = 0$ where $N = \{e_1, e_2, \ldots\}^\perp$ in H.

    Example: Let $I \in \mathcal{L}(L^2[0,1])$ be defined by $(If)(x) = \int_0^x f(t)dt$. Then If is Lipschitz and I is a compact operator. If π denotes projection onto the functions of mean zero, then $T = i\pi I \pi$ is a compact self-adjoint operator, and the spectral decomposition of T is given by $e_n(x) = \exp(2\pi nx)$, $n \in \mathbb{Z} - 0$, and $\lambda_n = 1/(2\pi n)$.

    Proof of the theorem: Choose f_n in H of norm one with $< Tf_n, f_n >\to \alpha$ where $\pm\alpha = ||T||$. By compactness we can assume f_n converges to a limit f; then $Tf = \alpha f$, and we may restrict to the subspace complementary to f and continue. By compactness this process terminates in countably many steps.

49. Now let T be an arbitrary bounded self-adjoint operator. The spectral theorem for T has 3 components.

    (A) The operator T can be expressed as a direct integral

    $$T = \int_{\mathbb{R}} \lambda d\pi_\lambda,$$

    where π_λ is an increasing family of projections.

    (B) The continuous functional calculus $C(\sigma(T)) \to \mathcal{L}(H)$ extends to the bounded Borel functions $L_b^\infty(\sigma(T))$ in a unique way, such that $f_n \to f$ monotonically implies $f_n(T) \to f(T)$ strongly. (Strong convergence means $T_n v \to Tv$ for all vectors v in H.)

    (C) There is a collection of measure μ_α on $\sigma(T)$ and an isomorphism $H \to \oplus L^2(\sigma(T), \mu_\alpha)$ such that the action of $C(\sigma(T))$ is realized by multiplication.

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All three results are closedly related; for example, we may take π_λ equal to the image of $\chi_{(-\infty,\lambda]}$ under the Borel functional calculus. Also (C) gives existence of the Borel functional calculus.

50. Theorem (Dominated Convergence): Let T_n be an increasing sequence of self-adjoint operators which is bounded above (e.g. $T_n \leq I$). Then there is an operator T such that $T_n \to T$ in the strong topology.

   Proof: Use the generalized Cauchy-Schwarz inequality, which states that for $T \geq 0$ we have $< Tv, w >^2 \leq < Tv, v >< Tw, w >$. Then if $||v|| = 1$, and $i > j$, we have

   $$||T_i v - T_j v||^4 = < (T_i-T_j)v, (T_i-T_j)v >^2 \leq < (T_i-T_j)v, v >< (T_i-T_j)^2 v, (T_i-T_j)v > .$$

   Since the operators T_i and T_j are bounded, the second term in the final product is bounded. By monotonicity and boundedness, the first term converges. Thus T_n converges strong; it is easy to see the limit is bounded and self-adjoint.

51. Theorem: The only algebra in $L_b^\infty(\mathbb{R})$ containing all bounded continuous functions and closed under monotone limits in the full algebra.

   These two theorems put together implies uniqueness of the Borel functional calculus, and can be used to prove its existence.

52. Theorem: For any bounded self-adjoint operator with spectrum contained in $[m, M]$, there is a unique family of projections π_λ such that $\pi_\lambda = 0$ for $\lambda < m$, $\pi_\lambda = I$ for $\lambda \geq M$, $\pi_\lambda \leq \pi_\mu$ for $\lambda \leq \mu$, $\pi_{\lambda+} = \pi_\lambda$ (that is, $\pi_\mu \to \pi_\lambda$ strongly as μ decreases to λ), such that

   $$T = \int_{\mathbb{R}} \lambda d\pi_\lambda.$$

   Here the integral converges in norm to T. That is, for any increasing sequence $\lambda_1 < \lambda_2 < \ldots < \lambda_n$, we have

   $$\sum \lambda_i(\pi_{\lambda_{i+1}} - \pi_{\lambda_i}) \leq T \leq \sum \lambda_{i+1}(\pi_{\lambda_{i+1}} - \pi_{\lambda_i}).$$

   (Note that if $|\lambda_{i+1} - \lambda_i| < C$ for all i, this last inequality implies T differs in norm by at most C from the Riemann sums.

Proof: For any operator S, define S^+ (using the functional) as $f(S)$, where $f(x) = (x + |x|)/2$. Then define π_λ to be projection onto the kernel of $(T - \lambda)^+ = T_\lambda$.

Since $T_\lambda \geq 0$, by generalized Cauchy Schwarz

$$< T_\lambda x, y >^2 \leq < T_\lambda x, x >< T_\lambda y, y >$$

so $T_\lambda x = 0$ if and only if $< T_\lambda x, x >= 0$. Since $\mu > \lambda$ implies $T_\mu \leq T_\lambda$, we have $\pi_\mu \leq \pi_\lambda$.

Semicontinuity follows easily.

53. Proof that $T = \int \lambda d\pi_\lambda$. The main point is to show that for $T = T^+ - T^-$, projection $\pi == \pi_0$ onto the kernel of T^+ satisfies (a) $T^+\pi = \pi T^+ = 0$ and (b) $T^-\pi = \pi T^- = T^-$.

The first part of (a) is obvious, and the second part follows from self-adjointness. For the first part of (b), note that $T^+T^- = 0$, so the image of T^- is contained in the kernel of T^+. Again the second part follows by self-adjointness.

Thus π and $I - \pi$ split H into invariant subspaces on which A is negative and positive. Continuing this process gives the direct integral.

54. Spectral measures. A vector v in H gives a non-negative measure μ_v on $\sigma(T)$ by $f \mapsto < f(T)v, v >$ for all f in $C(\sigma(T))$.

A *cyclic vector* v for T is one such that the closure of the span of $T^n v$, $n \geq 0$, is equal to H.

Theorem: A cyclic vector determines an isometric isomorphism $U : L^2(\sigma(T), \mu_v) \to H$ such that $U(\lambda f(\lambda)) = T(U(f))$.

Proof: Define a map $C(\sigma(T)) \to H$ by $f \mapsto f(T)v$. Then

$$||f(T)v||^2 = < f(T)v, f(T)v >= < f(T)^*f(T)v, v >= \int |f|^2 d\mu_v.$$

Thus U extends to an isometry, which is surjective because v is cyclic.

55. Theorem. If H is separable, there exists an (at most countable) sequence of vectors v_i, such that $H = \oplus H_i$ and v_i is cyclic in H_i.

Proof. Note that for any vector v, the closed span H_v of $T^n v$ and its complement are both left invariant by T. Now take a maximal set of vectors v such that $v \neq w$ implies H_v is perpendicular to H_w.

Corollary. If T is a self-adjoint operator on a separable Hilbert space, there exists an at most countable sequence of measures and an isomorphism $U : \oplus L^2(\sigma(T), \mu_n) \to H$ such that $U(\lambda f(\lambda)) = T(U(f))$.

56. Unitary operators and ergodic theory. A measure-preserving transformation gives a unitary operator U on L^2. Ergodicity is equivalent to no invariant vector other than the constants. Mixing is equivalent to the condition that U^n tends weakly to projection onto the constants. This convergence never holds in the strong topology, since $||U^n f|| = ||f||$ for all n.

Example: the baker's transformation is mixing, i.e. the shift on $\{0, 1\}^{\mathbb{Z}}$ with the binary measure. An irrational rotation of the circle is not mixing. An Anosov automorphism of a torus is mixing.

57. A *von Neumann algebra* $A \subset \mathcal{L}(H)$ is a $*$-algebra containing the identity and satisfying any of the equivalent conditions (a) $A = A''$, (b) A is closed in the weak operator topology, (c) A is closed in the strong operator topology. In particular, A is norm-closed, and hence a C^* algebra.

58. Let X be a standard Borel space, (e.g. $X = [0, 1]$), and let μ be a probability measure on X. Let $m : X \to \{0, 1, 2, \ldots, \infty\}$ be a Borel function (the *multiplicity*), and let $X' = \{(x, n) \in X \times \mathbb{N} : n \leq m(x)\}$; there is an natural projection $X' \to X$. Lift μ to a measure μ' on X' using counting measure on the fibers. Then $L^\infty(X, \mu)$ acts on $L^2(X', \mu')$ by multiplication. This is an example of a commutative von Neumann algebra.

59. Let $A \subset \mathcal{L}(H)$ be a *commutative* von Neumann algebra of operators on a separable Hilbert space.

Theorem. There is a measure μ on X, a multiplicity function m, and an isomorphism of H to $L^2(X', \mu')$ such that A becomes isomorphic to $L^\infty(X, \mu)$.

Proof. A contains a self-adjoint operator T such that $A = \{T\}''$, i.e. A is the smallest von Neumann algebra containing T. Then apply the

spectral decomposition and the Borel functional calculus to T.

In this sense the theory of commutative von Neumann algebras is equivalent to measure theory.

60. **Measurable dynamics and von Neumann algebras.**

Let Γ be a discrete group acting on a measure space V, such that for all $g \neq \mathrm{id}$ the set of points fixed by g has measure zero. Let $X = V/\Gamma$ be the space of orbits, $p : V \to X$ the projection. A *random operator* A_x assigns to each orbit x a linear operator on $\ell^2(p^{-1}(x))$, such that the function

$$(v, w) \mapsto\; < A_{p(v)} e_v, e_w >$$

is measurable on the set of (v, w) in $V \times V$ such that $p(v) = p(w)$. (Here e_v is a basis element for $\ell^2(\Gamma v)$.) Define $||A_x||$ to be the essential supremum of the usual norm on ℓ^2.

Theorem. The bounded random operators form a von Neumann algebra, whose center is isomorphic to the space of Γ-invariant bounded measurable functions.

In particular, if Γ acts ergodically we obtain a *factor*, that is a von Neumann algebra with trivial center.

61. The algebra A of random operators can also be described in terms of Hilbert space bundles. The Hilbert space $\ell^2(p^{-1}(x))$ is a bundle over the space X of orbits of Γ. This bundle is non-trivial; in fact, it has no sections (just as one cannot measurably pick a point in each orbit of an ergodic group action). Then A can be thought of as the space of *essentially bounded sections* (like the L^∞ functions) of the associated bundle $\mathcal{L}(\ell^2(p^{-1}(x)))$.

62. **Equivalence of projections.** Murray and von Neumann define two projections π, ρ to be equivalent if there is an operator T in the algebra A such that $TT^* = \pi$, $T^*T = \rho$.

For measurable dynamics, the natural projections come from measurable sets $E \subset V$, by projecting to $\ell^2(E \cap \Gamma v)$ for each orbit v.

Elements of Γ determine unitary operators in A. Two measurable sets E and F are *equivalent* if they can be partitioned into countably many disjoint pieces E_i, F_i such that $E_i = \gamma_i F_i$ for group elements γ_i.

To see this from an algebraic point of view, suppose $\pi = \oplus \pi_i$, $\rho = \oplus \rho_i$ and there are unitary U_i such that $U_i \pi_i = \rho_i U_i$. Then these two projections are equivalent, by considering $T = \sum U_i \pi_i$.

63. Types of factors. Let A be a factor in $\mathcal{L}(H)$, H a separable Hilbert space. A projection is *finite* if $\rho \sim \pi$, $\rho \leq \pi$ implies $\rho = \pi$.

Then there is a map D from projections into $[0, \infty]$, unique up to multiplication by $\lambda > 0$, such that $\pi \sim \rho$ if and only if $D(\pi) = D(\rho)$; $D(\pi) < \infty$ if π is finite; and $\pi = \rho_1 \oplus \rho_2$ implies $D(\pi) = D(\rho_1) + D(\rho_2)$. The possible images of D are:

$\{1, \ldots, n\}$; then A is type I_n.

$\{1, \ldots, \infty\}$; then A is type I_∞.

$[0, 1]$; then A is type II_1.

$[0, \infty]$; then A is type II_∞.

$\{0, \infty\}$; then A is type III.

64. Example: For $A = \mathcal{L}(H)$, H a separable Hilbert space, A is of type I_d where $d = \dim H$ (finite or infinite). To see this in the infinite dimensional case, note that any projection to a subspace H' isomorphic to H can be written as $\pi = TT^*$, where $T : H \to H'$ is an unitary isomorphism (a *partial isometry*); then T^*T is the identity.

65. When A is a factor, the equivalence classes of projections are totally ordered. Here is a proof in the case of an ergodic action by a single transformation f, and the projections coming from measurable sets.

Let E and F be sets of nonzero measure, and let n_1 be the least non-negative integer such that $f^{n_1}(E) \cap F > 0$ (meaning, has nonzero measure). Let E_1 and F_1 be the parts of E and F identified by f^{n_1}. Now choose n_2 so f^{n_2} gives the first overlap between $E - E_1$ and $F - F_1$. By construction, $n_2 > n_1$. Define E_2, F_2 as before, and continue. In the end, either $E = \bigcup E_i$ or $F = \bigcup F_i$, since $n_i \to \infty$. Each E_i is equivalent to F_i, so either $E < F$, $F < E$ or $E \sim F$.

66. The classification of projections generalizes the following theorem in measurable dynamics: either Γ has a single orbit (of cardinality $n < \infty$ or ∞), or Γ admits an invariant measure (finite or infinite), or any two sets of positive measure are equivalent.

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The classification of factors leads to the following results. If $\Gamma =< f >$ is the ergodic dynamical system associated to a single transformation, either f admits an invariant measure, or any two sets of positive measure are equivalent. In the former case, there are only four dynamical systems up to orbit equivalence, namely I_n, I_∞, II_1 and II_∞.