Analysis II Midterm: Solutions
Math 114 – Fall 2014

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function. Prove that for any $a < b$, we have $\int_a^b f'(x)\,dx \le f(b) - f(a)$. (You may assume $f'(x)$ exists a.e.)

   Proof. Define $f(x) = f(b)$ for $x > b$, and let $f_n(x) = n(f(x+1/n) - f(x))$. Since f is monotone increasing, $f_n(x) \ge 0$. Moreover $f'(x)$ exists a.e., and $f_n(x) \to f'(x)$ pointwise a.e. By Fatou's lemma this implies

   $$\begin{aligned} \int_a^b f' &= \int_a^b \lim f_n \le \liminf \int_a^b f_n \\ &= \liminf \left(n \int_b^{b+1/n} f \right) - \left(n \int_a^{a+1/n} f \right). \end{aligned}$$

   The last two terms represent the average of f over the intervals $[b, b+1/n]$ and $[a, a+1/n]$ respectively. By our convention, the first average is $f(b)$, and since f is increasing, the second average is at least $f(a)$. This gives $\int_a^b f' \le f(b) - f(a)$.

2. (i) Define the outer measure $m^*(E)$ of a set $E \subset [0,1]$.

   (ii) Prove there is a measurable $A \supset E$ such that $m(A) = m^*(E)$.

   (iii) Suppose $m^*(E) + m^*([0,1] - E) = 1$. Prove that E is measurable. (You may assume that sets of outer measure zero are measurable.)

   Answers. (i) $m^*(E) = \inf \sum_i \ell(I_i)$, where the infimum is taken over all countable collections of intervals $(I_1, I_2, \ldots)$ that cover E, and $\ell(I)$ denotes the length of I.

   (ii) By (i), for each $n > 0$ there exists an open set U_n of the form $U_n = \bigcup I_i$ such that $E \subset U_n$ and $m(U_n) \le m^*(E) + 1/n$. Then $A = \bigcap U_n$ is measurable, and it satisfies $m^*(E) \le m(A) \le \inf m(U_n) = m^*(E)$.

   (iii) By (ii) there are measurable sets $A \supset E$ and $B \supset E' = [0,1] - E$ such that $m(A) + m(B) = m^*(E) + m^*(E') = 1$. Since $A \cup B = [0,1]$, this implies

   $$m(A \cap B) = m(A) + m(B) - m(A \cup B) = 0.$$

   Now $(A - E)$ is a subset of $A \cap B$, so it too has measure zero; in particular, $(A - E)$ is measurable. Therefore $E = A - (A - E)$ is also measurable.

   (Note: this approach is much easier than trying to show that E satisfies Caratheodory's criterion, $m^*(A \cap E) + m^*(A \cap E') = m^*(A)$.)

3. Let $f : [a,b] \to \mathbb{R}$.

   (i) When does f have *bounded variation*? (Give the definition.)

   (ii) When is f *absolutely continuous*? (Give the definition.)

   (iii) Prove that if f is absolutely continuous, then f has bounded variation.

   Answers.

   (i) f has bounded variation if

   $$T(f, [a,b]) = \sup \sum_1^n |f(a_i) - f(a_{i-1})| < \infty,$$

   where the sup is over all partitions of $[a,b]$ into subintervals with endpoints $a = a_0 < a_1 < \cdots < a_n = b$.

(ii) f is absolutely continuous if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for any finite collection of *disjoint* intervals $(a_i, b_i) \subset [a, b]$ with $\sum_1^n |b_i - a_i| < \delta$, we have $\sum_1^n |f(b_i) - f(a_i)| < \epsilon$.

(iii) Since f is absolutely continuous, there is a $\delta > 0$ so the condition above holds for $\epsilon = 1$. Then $V(f, [c, d]) \leq 1$ whenever $|d - c| < \delta$. ,Choose points $a = a_0 < a_1 i < \cdots a_n = b$ that partition $[a, b]$ into intervals of length less than δ. Then

$$V(f, [a, b]) = \sum_1^n V(f, [a_i, a_{i-1}] \leq n,$$

so f has bounded variation.

4. State two precise theorems corresponding to two of Littlewood's three principles.

Answers.

(i) If E is a measurable set and $m(E) < \infty$, then for all $\epsilon > 0$ there exists a finite union of intervals J such that $m(E \triangle J) < \epsilon$.

(ii) For any measurable function $f : \mathbb{R} \to \mathbb{R}$ and $\epsilon > 0$, there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)$ outside a set of measure less than ϵ.

5. Mark each of the following assertions True (T) or False (F).

(a) ☐ If A is a subset of B then A cannot be an element of B.

False.

(b) ☐ If f is a measurable function and E is a measurable set, then $f^{-1}(E)$ is measurable.

False. (This would be true if E were a Borel set.)

(c) ☐ If f is a monotone function and E is a Borel set, then $f^{-1}(E)$ is Borel set.

True. (Because the preimage of an interval is an interval).

(d) ☐ If f is absolutely continuous on $[a, b]$, then $\int_a^b |f'(x)|\, dx \leq |f(b) - f(a)|$.

False. We might have $f(a) = f(b)$.

(e) ☐ Given a measurable function $f : \mathbb{R} \to [0, 1]$, there is a sequence of simple functions such that $\phi_n \to f$ uniformly on $\mathbb{R}$.

True. Pull back a finite partition of $[0, 1]$.

(f) ☐ If $E_1, E_2, \cdots$ are measurable subsets of $\mathbb{R}$, and $m(E_1)$ is finite, then $m(\bigcap E_n) = \lim m(E_n)$.

False.

(g) ☐ If $A \subset [0, 1]$ has measure zero then $A - A$ also has measure zero.

False. The Cantor set is a counterexample.

(h) ☐ If $f : \mathbb{R} \to \mathbb{R}$ is measurable and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $f \circ g$ is measurable.

False. The problem is that if A is measurable there is no guarantee that $g^{-1}(A)$ is measurable.

(i) ☐ If $f : [a, b] \to \mathbb{R}$ has bounded variation, and $|a_n - b_n| \to 0$, then $|f(a_n) - f(b_n)| \to 0$.

False. The map f might have jump discontinuities.

(j) ☐ Step functions are dense in $L^\infty[0, 1]$.

False. $f = \sum \chi_{[1/(n+1),1/(n+1)]}$, where the sum is over odd $n > 0$, cannot be approximated by a step function.